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Metatheory of Logics and the Characterization Problem

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1 Introduction

The word 'metatheory' denotes or perhaps suggests a theory of theories. Metascientific studies in the twentieth century used the term 'metatheory' to refer to investigations of theories in a variety of disciplines, for example, logic, sociology, psychology, history, etc. However, the philosophers of the Vienna Circle who made metatheoretical studies of science the main concern of their philosophy restricted metatheory to the logic of science modeled on developments in the foundations of mathematics. More specifically, the logic of science was intended to play a role similar to metamathematics in Hilbert's sense; that is, it was projected as formal analysis of scientific theories understood as well-defined linguistic items. The word 'metamathematics' was used before Hilbert, but with a different meaning from his (see Ritter et al. 1980: 1175–8). In the early nineteenth century, mathematicians, like Gauss, spoke about metamathematics in an explicitly pejorative sense. It was for them a speculative way of looking at mathematics – a sort of metaphysics of mathematics. A negative attitude to metaphysics was at that time inherited from Kant and early positivists. The only one serious use of 'metamathematics' was restricted to metageometry, and that was due to the fact that the invention of different geometries in the nineteenth century stimulated comparative studies. For example, investigations were undertaken of particular axiomatizations, their mutual relations, models of various geometrical systems, and attempts to prove their consistency. The prefix 'meta' presently suggests two things. First, it indicates that metatheoretical considerations appear 'after' (in the genetic sense) theories are formulated. Secondly, the prefix 'meta' suggests that every metatheory is 'above' a theory which is the subject of its investigations. It is important to see that 'above' does not function as an evaluation but only indicates the fact that metatheories operate on another level than theories do. A simple mark of this fact consists in the fact that theories are formulated in an object language, and metatheories are expressed in a related metalanguage.

It is probably not accidental that Hilbert passed to metamathematics through his famous study of geometry and its foundations. Hilbert projected metamathematics as a rigorous study of mathematical theories by mathematical methods. Moreover, the Hilbertian metamathemics, due to his views in the philosophy of mathematics (formalism) was restricted to finitary methods. If we reject this limitation, metamathematics can be described as the study of mathematical systems by mathematical methods; they cover those that are admitted in ordinary mathematics, including infinitistic or infinitary – for instance, the axiom of choice or transfinite induction. However, this description is still too narrow. Hilbert's position in metamathematics can be described as follows: only syntactic combinatorial methods are admissible in metatheoretical studies. However, the semantics of mathematical systems is another branch of the methatheory of mathematics. It is interesting that the borderline between syntax and semantics corresponds to some extent with the division between finitary and infinitary methods. I say 'to some extent' because we have also systems with infinitely long formulas (infinitary logic). It is clear that the syntax of infinitary logics must be investigated by methods going beyond finitary tools. It was also not accidental that systematic formal semantics (model theory) which requires infinitistic methods appeared in works by Alfred Tarski who, due to the scientific ideology of the Polish mathematical school, was not restricted to the dogma that only finite combinatorial methods are admissible in metamathematics. Today, metamathematics can be divided into three wide areas: proof theory (roughly speaking, it corresponds to metamathematics in Hilbert's sense if proof-methods are restricted to finitary tools, or it is an extension of Hilbert's position if the above-mentioned restriction is ignored), recursion theory (which is closely related to the decision problem, that is, the problem of the existence of combinatorial procedure providing a method of deciding whether a given formula is or is not a theorem) and model theory, that is, studies of relations between formal systems and structures which are their realizations; model theory has many affinities with universal algebra.

The metatheory of logics (plural is proper, because we have many competing logical systems) is understood here as a part of metamathematics restricted to logical systems. We can also use the word 'metalogic' and say that it refers to studies of logical systems by mathematical methods. This word also appeared in the nineteenth century (see Ritter et al. 1980: 1172–4), although its roots go back to the Middle Ages (*Metalogicus* of John of Salisbury). Philosophers, mainly neo-Kantians, understood metalogic to be concerned with general considerations about logic. The term 'metalogic' in its modern sense was used for the first time in Poland (by Jan Łukasiewicz and Alfred Tarski) as a label for the metamathematics of the propositional calculus. Thus, metalogic is metamathematics restricted to logic, and it covers proof theory, investigations concerning the decidability problem, and model theory with respect to logic.

When we say that metalogic is a part of metamathematics, it can suggest that the borderline between logic and mathematics can be sharply outlined. However, questions like 'What is logic?' or 'What is the scope of logic?' have no uniformly determined answer. We can distinguish at least three relevant subproblems that throw light on debates about the nature of logic and its scope. The first issue focuses on the so called first-order thesis. According to this standpoint, logic should be restricted to standard first-order logic. The opposite view contends that the scope of logic should be extended to a variety of other systems, including, for instance, higher-order logic or infinitary logic. The second issue focuses on the question of rivalry between various logics. The typical way of discussing the issue consists in the following question: Can we or should we replace classical logic by some other system, for instance, intuitionistic, many-valued, relevant or paraconsistent logic? This way of stating the problem distinguishes classical logic as the system which serves as the point of reference. Thus, alternative or rival logics are identified as non-classical. There are two reasons to regard classical logics as having a special status. One reason is that classical logic appeared as the first stage in the development of logic; it is a historical and purely descriptive circumstance. The second motive is clearly evaluative in its character and consists in saying that classical logic has the most 'elegant' properties or that its service for science, in particular, for mathematics, is 'the best.' For example, it is said that abandoning the principle of excluded middle (intuitionistic logic), introducing more than two logical values (many-valued logic), changing the meaning of implication (relevant logic) or tolerating inconsistencies (paraconsistent logic) is something wrong. It is also said that some non-classical logics, for example, intuitionistic or many-valued logics, considerably restrict the applicability of logic to mathematics. It is perhaps most dramatic in the case of intuitionistic logic, because it or other constructivistic logics lead to eliminating a considerable part of classical mathematics. Thus, this argument says that only classical (bivalent or two-valued) logic adequately displays the proof methods of ordinary mathematics. While the discussion is conducted in descriptive language, it appeals to intuitions and evaluations of what is good or wrong in mathematics. The situation is similar as far as the matter concerns metalogical properties of particular systems such as completeness, decidability or the like, because it is not always obvious what it means to say that a logic possesses them 'more elegantly' than a rival system. The priority of classical logic is sometimes explained by pointing out that some properties of non-classical logic are provable only classically. This is particularly well-illustrated by the case of the completeness of intuitionistic logic: Is the completeness theorem for this logic intuitionistically provable? The answer is not clear, because the stock of intuitionistically or constructively admissible methods is not univocally determined, and they vary from one author to another. Finally, our main problem (what is logic and what is its scope?) is also connected with extensions of logics. If we construct modal logics, deontic logics, epistemic logics, etc., we usually start with some basic (propositional or predicate) logic. We have modal propositional or predicate systems which are based on classical, intuitionistic, many-valued or some other basic logic. Does any given extension (roughly speaking, an extension of a logic arises when we add new concepts, for example necessity, to old ones in such a way that all theorems of the system before extension are theorems the new system) of a chosen basic logic preserve its classification as a genuine logic or does it produce an extralogical theory? The *a priori* answer is not clear, even when we decide that this or that basic system is *the* logic. The problem of the status of extensions of logic is particularly important for philosophical logic because it consists mainly of systems of this sort.

The three issues concerning the question 'What is logic?' are mutually interconnected. The choice between first-order logic or higher-order logic automatically leads to the two other issues, because it equally arises with respect to any alternative logic and any extension of a preferred basic logic. Thus, we have a fairly complex situation. Yet the above division into three issues does not exhaust all problems. Usually it is assumed that first-order logic (classical or not) is based on the assumption that its universe it not empty. However, as Bertrand Russell once remarked, that it is a defect of logical purity, if one can infer from the picture of logic that something exists. This is perhaps the main motivation for so-called free logic, that is, logic without existential assumptions (logic admitting empty domains). Is it classical or not? The described situation suggests a pessimism as far as the matter concerns a natural and purely descriptive characterization of logic; it seems that an element of a convention is unavoidable here. A further reason that the domain of metalogic cannot be sharply delimited is that several metalogical or metamathematical results distinguish logical (even in a wider sense) from other formal systems. Assume that we decide to stay with the first-order thesis. The second Gödel theorem (the unprovability of the consistency of elementary arithmetic) clearly separates pure quantification logic from formal number theory. It is one reason that metamathematical results are of interest for metalogic. Metalogical investigations also use several concepts that are defined in general metamathematics, for example formal system, axiomatizability, consistency, completeness, provability, etc. Fortunately, we are not forced to answer the borderline question in a final manner. My aim in this essay is to review the most essential metalogical concepts. Classical first-order logic is taken as the paradigm. The treatment is rather elementary. Although I assume some familiarity with syntax and semantics of first-order logic as well as with several simple concepts of set theory, most employed concepts are explained. However, some important concepts of metalogic, for instance that of recursive function, do not allow a brief and elementary clarification. On the other hand, it would be difficult and not reasonable to resign from them. These concepts are marked by * and the reader is asked to consult textbooks listed in the references. In particular, I recommend Hunter (1971) and Grzegorczyk (1974); moreover, Pogorzelski (1994) is suggested as the fullest survey of metalogic (I follow this book in many matters). Special attention will be given to relations between syntactic and semantic concepts that are most strikingly displayed by (semantic) completeness theorems. I do not enter into historical details, although it seem to be proper to include dates when some fundamental theorems were proved (references to original papers are easily to be found in works listed in the Bibliography).

The characterization problem is a special metalogical issue. It consists in giving sufficient and necessary conditions which determine particular logics or classes of logics. These conditions can be syntactic, semantic, or mixed. Let me explain the problem in the case of the propositional calculus. It has been axiomatized in various ways. However, one axiomatic base, rather long, is particularly convenient here. The axioms are these (I use the Hilbert-style formalization use of axiom-schemata. Thus, the letters A, B, C are metalinguistic variables referring to arbitrary formulas of the propositional calculus and modus ponens (B is derivable from A and $A \to B$) is the only inference rule:

(A1) $A \to (B \to A)$

(A2) $(A \to (A \to B)) \to (A \to B)$

(A3) $(A \to B) \to ((B \to C) \to (A \to C))$

(A4) $A \wedge B \to A$

(A5) $A \wedge B \to B$

(A6) $(A \to B) \to ((A \to C) \to (A \to B \wedge C))$

(A7) $A \to A \vee B$

(A8) $B \rightarrow A \vee B$

(A9) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

(A10) $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$

(A11) $(A \leftrightarrow B) \rightarrow (B \rightarrow A)$

(A12) $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$

(A13) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

(A14) $A \rightarrow \neg\neg A$

(A15) $\neg\neg A \rightarrow A$

A nice feature of this set of axioms is that we can easily distinguish subsets related to particular connectives. (A1)–(A3) characterize implication, (A4)–(A6) conjunction, (A7)–(A9) disjunction, (A10)–(A12) equivalence, and (A13)–(A15) negation. Now if we eliminate (A15), we obtain the axiom set for intuitionistic logic. Thus, we can say that (A1)–(A15) solve the characterization problem for classical propositional logic, but (A1)–(A14) do the same job for intuitionistic propositional logic, provided that the characterization problem is to be solved by axiomatic methods. Other ways of characterizing logical systems proceed by matrices (truth-tables), semantic tableaux*, trees*, semantic games* or Hintikka sets*, but all provide conditions which separate various more or less alternative logics. One characterization result recently became particularly famous. It is the celebrated Lindström theorem which establishes very general conditions for first-order logic. This theorem and will be presented in a separate section below.

Formal metalogical results are interesting in themselves as well as being philosophically important. The problem of the nature of logic has a decisively philosophical character. Several accepted intuitions about logic have gained widespread acceptance: that logic is formal, universal or topic-neutral, and provides sound (leading always from truths to other truths) rules of inference. It is interesting to look at metalogical results as capturing old intuitions; for example, that expressed in the following words of Petrus Hispanus: *dialectica est art artium et scientia scientiarum ad omnium scientiarun methodorum principia viam habent* (dialectics (that is, logic) is the art of arts and the science of sciences that provides methodological principles for all sciences). Another illustration of the philosophical importance of formal results is that, according to intuitionism, intuitionistic and classical logic are simply incomparable. It is sometimes maintained that differences between alternative logics consist in the assignment of different meanings to logical constants and facts, and that these systems are not intertranslatable. As the characterization problem displayed by axioms for the propositional calculus shows, however, at least from the classical point of view, classical logic and all weaker systems are perfectly comparable.

2 Logic via Consequence Operation and Semantics

Intuitively speaking, logic provides manuals for proving some propositions on the basis of some assumptions. These manuals consist of inference rules; for example, *modus ponens* instructs us that we may logically pass from A and $A \rightarrow B$ as premises to B as conclusion. Assume that **R** is a set of inference rules. The notation $X \vdash^{R} A$ expresses the fact that a formula A is provable (derivable) from the set X of assumptions, relative to rules of inference from **R** (I will omit the superscript indexing the provability sign in due course). We define

(DCn) $A \in Cn(X) \Leftrightarrow X \vdash A$.

Although *Cn* (the consequence operation) and $\vdash$ (the consequence operator) are mutually interdefinable, there is a categorial difference between them. Let **L** be a language understood as a set of formulas. *Cn* is a mapping from 2^{L} to 2^{L} that transforms sets of formulas into sets of formulas, and the consequence operator maps 2^{L} into **L**, that is, sets of formulas are transformed into single formulas.

The analysis of logic *via* the consequence operator is much more common than that using *Cn* (see Segerberg (1982) for the first approach). It is also more closely related to codifications of logic *via* natural deduction techniques or sequents which are also used (see Hacking 1979) in analyzing the concept of logic. I will take another route, however, and concentrate on the consequence operation (I follow Surma (1981); see also Surma (1994)). The first question that arises here is this: how many consequence operations do have we? The answer is that there are infinitely many *Cn*'s. Thus, we need to establish some constraints selecting a 'reasonable' consequence operation (or operations). Tarski characterized the classical axiomatically *Cn* (in fact, Tarski axiomatized the consequence operation associated with the propositional calculus; the axioms given below concern the consequence operation suitable for first order logic). The axioms are these (explanations of symbols: $\varnothing$, the empty set; **L**, language; $\mathcal{N}_0$, the cardinality of the set of natural numbers; $\subseteq$, inclusion between sets; $\in$, the membership relation (being an element of a set); **FIN**, the class of all finite sets; $\cup$, union of sets; $\{A\}$, the set consisting of A as the sole element; $\cap$, product of sets; /, the operation of substitution for terms):

(C1) $\varnothing \leq \mathbf{L} \leq \mathcal{N}_0$

(C2) $X \subseteq CnX$

(C3) $X \subseteq Y \Rightarrow CnX \subseteq CnY$

(C4) $CnCnX = CnX$

(C5) $A \in CnX \Rightarrow \exists Y \subseteq X \wedge Y \in \mathbf{FIN}(A \in CnY)$

(C6) $B \in Cn(X \cup \{A\}) \Rightarrow (A \rightarrow B) \in CnX$

(C7) $(A \rightarrow B) \in CnX \Rightarrow B \in Cn(X \cup \{A\})$

(C8) $Cn\{A, \neg A\} = \mathbf{L}$

(C9) $Cn\{A\} \cap Cn\{\neg A\} = \varnothing$

(C10) $A(\mathrm{v/t}) \in Cn\{\forall v A(v)\}$, if the term t is substitutable for v.

(C11) $A \in CnX \Rightarrow \forall v A(v) \in CnX$, if v is not free* in X, for every $B \in X$.

We can divide the axioms (C1–C11) into three groups. The first group includes (C1–C5) as general axioms for *Cn*. (C1) says that the cardinality of **L** is at most denumerably

(denumerably – finitely or so many as natural numbers) infinite, (C2) that any set is a subset of the set of its consequences, (C3) established the monotonicity of *Cn* (in general, a function *f* is monotonic if and only if $x \leq y$ entails $fx \leq fy$; in fact, inclusion is a kind of the ≤-operation), (C4) its idempotency (a function *f* is idempotent if and only if $ffx = fx$), (C5) states the finiteness condition which means that if something belongs to $Cn(X)$, it may be derived from a finite subset of *X*. In other words: every inference is finitary, that is, performable on the base of a finite set of premises and, according to the character of rules, finitely long. It is an important property, because there are also infinitary logical rules, for example the ω-rule which leads (roughly speaking) from the infinite sequence of premises $P(1)$, $P(2)$, $P(3)$, . . . to the conclusion $\forall nP(n)$, but it is commonly recognized that human beings cannot effectively use such rules. (C1–C5) do not provide any logic in its usual sense. The logical machinery is encapsulated by the rest of axioms (related to logic based on negation, implication, and the universal quantifier): (C6) is modus ponens (it shows that modus ponens is the inverse of the deduction theorem), (C7) the deduction theorem (if *B* is derivable from the set *X* plus *A*, then the implication $A \rightarrow B$ is derivable from *X*; if it is to be applied to predicate logic, we must assume that *A* and *B* are closed formulas, that is formulas without free variables), (C8)–(C9) characterize negation, and (C10–C11) characterize the universal quantifier. We can also add axioms suitable for identity or introduce the consequence operation for intuitionistic logic.

Logic (more precisely: classical first-order logic) can be defined as $Cn\varnothing$. More formally we have:

(DL1) $A \in \mathbf{LOG} \Leftrightarrow A \in Cn\varnothing$, or, equivalently $\mathbf{LOG} = Cn\varnothing$.

Of course, modifications of *Cn* in accord with the ideas of alternative logics lead to their related definitions. For example, intuitionistic logic is given by the equality $\mathbf{LOG_i} = Cn_i\varnothing$. (DL1) looks artificial at first sight, because it is clear that the logical content is related to axioms imposed on *Cn*; clearly, the empty set here is a convenient metaphor: we can derive something from the empty set only because of the logical machinery already built into *Cn*. Hence, we have the problem of deciding what stipulations about the consequence operation are proper for logic. This question concerns general as well as special axioms. Worries concerning which logic, classical or some alternative, is the 'logic' also remain on this approach; for example, we can consider this question with respect to modal extensions or formal systems which contain rules related to axioms of arithmetic. Are $Cn_m\varnothing$ (the set of modal consequences, relatively to a system of modal logic, of the empty set) or $Cn_{ar}\varnothing$ (the set of arithmetical consequences of the empty set) logics)? As far as the general axioms are concerned, we can, for instance, drop the requirement of monotonicity (it leads to non-monotonic logics used in computer science) or finiteness (infinitary logic). Hence, any definition of logic *via* the consequence operation needs an additional justification. I will present a motivation for classical logic which can be easily applied to other systems.

First of all, let us observe that (DL1) is equivalent to two other statements, namely (an explanation concerning (DL3): an operation *o* closes the set *X* if and only if $oX \subseteq X$, that is, applications of *o* to *X* do not produce elements which do not belong to *X*):

(DL2) $A \in$ **LOG** if and only if $\neg A$ is inconsistent.
(DL3) **LOG** is the only non-empty product of all deductive systems (theories), that is, sets which satisfy the condition: $CnX \subseteq X$ (are closed under Cn).

Now, (DL2) and (DL3) surely define properties which we expected to be possessed by any logic. We agree that negations of logical principles are inconsistencies and that logic is the common part of all, even mutually, inconsistent theories. Additionally, (DL3) entails that logical laws are derivable from arbitrary premises. Thus, we have the equivalence: $A \in Cn\varnothing$ if and only if $A \in CnX$, for any X, and the equality **LOG** = $Cn\varnothing = CnX$, for any X. These considerations show that (DL1) and its equivalents express an important intuition, namely that logic is universal in the sense that it does not require any premises, or is deducible from arbitrary assumptions.

Yet one might argue that such a construction of logic is circular because it defines logic by means of the prior assumption that something is logical. This objection can be easily met by pointing out that our definitions are inductive, that is, selects logical axioms as so called initial conditions and then shows how inductive conditions (in fact, the rules of inference coded by Cn) lead step by step to new logical elements. On the other hand, it is perhaps important for philosophical reasons to look at an independent characterization of logic. This is provided by semantics and it is expressed by (a model of a set X of sentences is a structure consisting of a universe of objects and a collection of relations defined on the universe such that all sentences belonging to X are true; if we admit open formulas, that is, formulas with free variables, a model of a set X of formulas is a structure in which all formulas belonging to X are satisfied):

(DL4) $A \in$ **LOG** if and only if for every model **M.**, A is true in **M**.

This last definition describes logic as universal in the sense that logical laws are true in every model (domain). It is related to the old intuition that logic is topic neutral, that is, true or valid with respect to any particular subject matter. Intuitively, there is an obvious link between (DL1)–(DL3) and (DL4). However, we have no formal tools that prove all these definitions are equivalent. Since (DL1)–(DL3) are syntactical descriptions of logic (they use the concepts of consequence operation or consistency which are just syntactic), but (DL4) is semantic in its essence (it defines logic *via* the concept of a model), any comparison of the two approaches requires a rigorous investigation of how syntax and semantics are related. In fact, it consists in a comparison of the set of theorems (the set of provable formulas) of a system under investigation with the set of its validities (truths, tautologies).

3 Metalogic, Syntax and Semantics

Although we basically intend to achieve a precise comparison of syntax and semantics in logic, this section provides an opportunity to introduce several important metalogical concepts and properties (others will be defined in the next section). Let **S** be an arbitrary formal system formulated in a language **L**. The most important metalogical concepts are summarized by the following list:

S is consistent if and only if $Cn\mathbf{S} \neq \mathbf{L}$; if **S** contains the negation sign this definition is equivalent to the more standard: **S** is consistent if and only if no inconsistent pair (that is, consisting of A and $\neg A$) of formulas belongs to the consequences of **S**.

S is Post-complete (the name honors of Emil Post, an American logician who defined the property in question) if and only if $Cn(\mathbf{S} \cup \{A\}) = \mathbf{L}$, for any formula A which is not a theorem of **S**.

S is syntactically complete if and only if for any A, either $A \in Cn\mathbf{S}$ or $\neg A \in Cn\mathbf{S}$.

S is semantically complete if and only if every provable formula of **S** is true in every model of **S** and every validity (truth) of **S** is provable in it.

S is decidable if and only if the set of its theorems is recursive*.

S is axiomatizable if and only if there is a set $\mathbf{Ax} \subseteq \mathbf{S}$ such that $\mathbf{S} = Cn\mathbf{Ax}$; if **Ax** is finite (recursive) we say that **S** is finitely (recursively) axiomatizable.

Some comments are in order. Various labels for particular properties are employed by various authors. For example, syntactical completeness is sometimes called negation-completeness. Semantic completeness has in fact two ingredients. The direction from provability to validity (every truth is provable) is considered as soundness (correctness, adequacy) and semantic completeness proper, so to speak, is expressed by the reverse implication (every truth is provable). The given definition of decidability is related to the Church thesis*, that is, the proposal to identify intuitively calculable functions (calculable in the finite mechanically performable steps) with recursive functions. Finally, the definition of axiomatizability does not exclude the situation that **S** forms its own axiomatic base.

We are mainly interested in properties of logic. The propositional calculus is consistent, post-complete, syntactically incomplete (it is enough to consider a single variable; neither p nor $\neg p$ are theorems of propositional logic), sematically complete, decidable (by the truth-table method) and finitely axiomatizable (by concrete formulas) or recursively axiomatizable (by schemata). One qualification is needed with respect to the concept of post-completeness. This property holds for the propositional calculus with axioms as concrete formulas and the rule of substitution. Fortunately, we can define another property, parallel to post-completeness which is possessed by the propositional calculus when it is formalized by axiom-schemata. Now, first-order predicate logic is consistent, not post-complete (if we add, for example, the sentence ‘there are exactly two objects’ which is not a logical theorem as a new axiom, the resulting system is not inconsistent), not syntactically complete, semantically complete (proved by Kurt Gödel in 1929), undecidable (proved by Alonzo Church in 1936), and finitely or recursively axiomatizable. All these facts apply to first-order predicate logic with identity. Gödel proved in 1931 two famous theorems (both of which assume that arithmetic is consistent): (1) every formal system strong enough for the elementary arithmetic of natural numbers is syntactically incomplete; (2) the consistency of arithmetic is unprovable in arithmetic; both theorems assume that arithmetic is consistent. The first theorem implies that arithmetic is not recursively axiomatizable. Tarski showed in 1933 that the set of arithmetical truths is not definable arithmetically*. Finally, Church proved in 1936 that arithmetic is undecidable. These four theorems are usually called limitative theorems, because they point out limitations inherent to any formalism sufficiently rich to cover the arithmetic of natural numbers.

For our aims, semantic completeness is the most important. In its most general form, the completeness theorem (in its strong form) says (the symbol $\models$ stands for validity):

(CT) **S** is semantically complete if and only if: $\mathbf{S} \vdash A \Leftrightarrow \mathbf{S} \models A$.

(CT) is equivalent to the Gödel–Malcev theorem:

(GM) **S** is consistent if and only if it has a model.

The proof of (GM) requires the axiom of choice* (or its equivalents) which means that it is not a constructive theorem. The most popular proof of (CT) uses the Lindenbaum lemma: every consistent set of sentences has a maximal consistent extension (maximality means here that adding any sentence to a maximally coinsistent set leads to inconsistency); this lemma is also not constructive. If we put $\varnothing$ instead **S** in (CT), we obtain $\varnothing \vdash A$ if and only if $\varnothing \models A$. By (DCn), it gives the weak completeness theorem

(CT1) $A \in Cn\varnothing \Leftrightarrow \varnothing \models A$.

Since the right part of (CT1) expresses the fact that A is true in all models, it legitimizes the equivalence of (DL1) and (DL4) for first-order predicate logic with identity. It should be clearly noted that the completeness theorem, although it establishes the parity of syntax and semantics in semantically complete systems, it does not provide in itself any definition of logic. However, if we agree that universality is its characteristic property, (CT1) shows that universality in the syntactic sense (provability from the empty set of premises) is exactly equivalent to universality in the semantic sense (truth in all models or logical validity). Moreover, this part of (CT) (or (CT1)) which expresses the soundness property (if a formula is provable, it is also true) justifies the intuition that logical rules are infallible: they never lead from truths to falsehoods.

The universality property is also displayed by another theorem, the neutrality theorem, which asserts that first-order predicate logic with identity does not distinguish any extralogical concept, that is, any individual constant or predicate parameter (c_i, c_j are individual constants, P_k, P_n are predicate parameters, the notation $A(c)$ and $A(P)$ means that a constant c (predicate parameter P) occurs in A):

(N) (a) $A(c_i) \in \mathbf{LOG} \Rightarrow A(c_j/c_i \in \mathbf{LOG}$; (b) $A(P_k) \in \mathbf{LOG} \Rightarrow A(P_n/P_k) \in \mathbf{LOG}$.

This theorem says that if something can be provable in logic about an object or its property, the same can be also proved about any other object or property. It is of course another aspect of the topic-neutrality of logic.

4 The Characterization Problem for First-order Logic

The strong completeness theorem motivates a stronger understanding of logic. Let **T** be an extralogical theory (axiomatized by the axioms belonging to the set **Ax**). Thus **T** is the ordered triple (see Rasiowa and Sikorski 1970: 187).

⟨**L**, *Cn*, **Ax**⟩.

Now the consequence operation *Cn* operating on **L** and **Ax** generates the logic of **T**. Denote logic in this extended sense by $\mathbf{LOG_T}$ and logic given by (DL1) by $\mathbf{LOG}_\varnothing$ (it operates on the empty set). Of course, $\mathbf{LOG}_\varnothing \subseteq \mathbf{LOG_T}$. The modification is not essential for logic in this sense: the stock of logical rules, given by $Cn\varnothing$, is the same. However, this extended concept of logic, which focuses on its applications, leads to a more general formulation of the characterization problem.

Call a logic regular if its logical symbols obey classical (Boolean) principles (it is practically restricted to negation; roughly speaking, 'Boolean' means that our logic is perfectly two-valued); we also assume that **L** is countable, that is, contains at most denumerably many sentences. A logic satisfies the compactness property (Com) if and only if it has a model if its every finite subset has a model. A logic satisfies the Löwenheim–Skolem property (LS) if and only if it has a countable model if it has an infinite model. The Lindström theorem (proved by Per Lindström in 1969) is the statement:

(L) First-order predicate logic is the strongest logic which satisfies (Com) or (CT), and (LS).

For example, second-order logic (first-order logic has quantifiers ranging over individuals; second-order logic also admits quantification over properties – the sentence 'for any object *x*, there is a property *P* such that *x* has *P*' is an example of a second-order sentence) satisfies neither (Com) nor (LS), but (CT) holds for it, if we admit second-order quantification over special entities, and logic with the quantifier 'there are uncountably many' is complete, but then it does not obey (LS). Of course, (L) holds also for logic defined by (DL1), that is, for $Cn\varnothing$. Let me add that no counterparts of (L) are known with respect to non-classical logics, in particular, intutionistic or many-valued logics. The reason is that they are not regular.

There is a considerable debate concerning the interpretation and consequences of (L) (see Barwise 1985). All parties agree that (L) asserts the limitations on the expressive power of first-order predicate logic. In particular, several mathematical concepts, like finiteness, cannot be defined in its language. Hovever, it is a matter of controversy whether (L) determines that only first-order predicate logic deserves to be counted as *the* logic. The first-order thesis, previously explained, restricts the scope of logic to first-order logic, but the opposite standpoint maintains that if logic is to serve mathematics, its expressive power must be much greater than that of first-order languages. It is now clear why this problem becomes central when an extended concept of logic is assumed. Since definability is traditionally regarded as a logical issue, its limitations are perceived as limitations of the power of logic. I will come back to these questions in the next (and final) section.

5 Final Remarks

In this section, I come back to philosophical issues concerning the concept of logic. Let me start with the first-order thesis. Its opponents argue that it restricts the application of logic in science, in particular, in mathematics, which requires that logic should have a considerable expressive or defining power in order to capture various mathematical concepts. On the other hand, the first-order thesis focuses on the universality property of logic and the infallibility of its inferential machinery (see Woleński 1999). Thus, we have to do here with a conflict between two different expectations concerning logic. The postulate that logic should have great expressive power recalls the ambitious projects of *logica magna* or *lingua characteristica* proposed by Leibniz, Frege, or Russell and intended as languages which are able to cover the whole of science or at least mathematics. The first-order thesis motivated by (L) and (N) sees logic as providing universally valid theorems, being the common part of all deductive systems, always generating a perfectly sound inference machinery. The issue is serious because either we can have strict universality or languages with a great expressive power, but not both virtues together. We can assume that $Cn\varnothing$ always provides secure rules of inference. Thus, the point is what should be regarded as logical: only propositional connectives, quantifiers, and identity, or perhaps also other concepts, like finiteness. It is not surprising that (CT) contributes to our understanding of the universality of logic. However, it was not expected that (Com) and (LS) do too, though if first-order predicate logic does not distinguish any extralogical concepts, it also should be neutral with respect to the cardinality of models, that is, the number of elements in their universes. It is interesting that there are also problems when we consider identity as a logical concept. The argument for its status as a logical constant stems from the fact that first-order logic with identity relation satisfies (CT), (N), and (L). On the other hand, identity enables us to define numerical quantifiers, for example, ‘there are exactly two objects’, but there are doubts whether such phrases deserve to be called logical. Thus we have reasons to say that the prospects for an answer to the question ‘What is logic?’ that is unconditional and free of at least some degree or arbitrariness, are not encouraging. The problem becomes still more complicated when non-classical logics are taken into account.

New problems arise when extensions of a basic logic are analyzed. It may be demonstrated by modal logic. Since modal systems are more closely treated in a separate chapter in this *Companion*, I limit myself to a very sketchy remarks. We can and even should ask whether $\Box$ (necessity) and $\Diamond$ (possibility) are logical constants? One might argue that since special conditions, related to particular modal systems, are imposed on modal models, especially on so-called accessibility relations (for example, deontic logic requires that this relation is irreflexive, the system **T** is associated with the condition of symmetry, etc.), modal logics are not universal. On the other hand, the system **K** does not require any particular constraint. Yet we can say that its characteristic formula $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ is a modal translation of the theorem of first-order logic $\forall x(A \rightarrow B) \rightarrow (\forall xA \rightarrow \forall xB)$. However, **K** is a very weak system and does not display all traditional intuitions concerning logical relations between modalities. Thus, we perhaps should decide: either universality (no special provisos on modal models) or more content, like in the case of the controversy over the first-order thesis.

How then does metalogic contribute to our understanding of logic? The answer seems to be this. Although metalogical theorems do not provide answers which are free of conventional elements, they precisely show those points where intuitions go beyond formal results.

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