Impredicativity and Schematic Generality

A definition is said to be impredicative if it quantifies over a totality to which the *definiendum* belongs. A long-standing philosophical debate has focused on impredicativity as a source of semantic instability ([8]), as a form of circularity ([11], [3]) and, more recently, as a violation of the potential nature of the infinite domains usually involved in this kind of definitions ([6]).

In this talk, I will examine the phenomenon of impredicativity in the light of different accounts of quantification, in order to test the hypothesis that the semantic instability and the circularity usually attributed to it arise only in virtue of the meaning of quantification in classical logic, as the infinite conjunction (or disjunction) of its instances, and of the corresponding notion of generality involved. In this framework, circularity arises because not only the *definiendum* of an impredicative definition is one of these instances – namely one of the possible values of the variables bound by the quantifier – but because the classical meaning of the quantification requires an exhaustive examination of all of these instances.

Three alternative approaches to impredicativity, based on different interpretations of the quantification, will be explored and compared. These are inspired respectively by original insights of Weyl, Carnap and Russell, and have been recently rediscovered, in the light of current logical developments. Such approaches share an explanation of the generality involved in the quantification that is currently defined *generic*, i.e. not instance-based ([5]), and allow to save syntactically impredicative definititions from the ban included in the traditional reading of Russell's Vicious Circle Principle (VCP, [11]).[1] More precisely, they allow the VCP itself – and especially the notion of totality involved – to be relativised with respect to the classical meaning of quantification. Despite their similar effects on the phenomenon of the impredicativity, different motivations will be identified, justifying different non-classical formalisations and ultimately revealing a different notion of (even generic) generality.

The first non-classical treatment of impredicativity was proposed by Weyl ([13]) and is based on the adoption of intuitionistic logic, in which the truth of universal statements does not depend on the verification of their instances – impossible in the case of infinite domains – but "lies in the essence" shared by all of them. A recent implementation of this approach has been formalised in semi-intuitionistic logic ([5], [6]). Also in this case, the universal quantification expresses something stronger than the absence of counterexamples (in that it is not dual to the existential quantifier), relying instead on fully general facts about the properties involved in the generalisation. Such an approach is particularly useful in the case of a potentialist framework, because it makes the universal generalisation available from the beginning of the generative process

[1] "No totality can contain members defined in terms of itself".

of the instances, and independently of the stages.

A competing approach follows a Carnapian insight ([1]) and justifies impredicativity on the basis of what he called the "specific generality" (as opposed to "numerical generality") of the quantification involved, whose behaviour is independent of running through all the individual cases, but relies on the uniformity of the proofs of the corresponding universal statements. The generality pointed out by Carnap anticipated what is currently called "schematic generality" ([2]) and is usually attributed to parameters. This approach suggests a constructivist program and supports the impredicative developments of type theory in the polymorphic lambda calculus ([7], [4]). In this framework, impredicativity is allowed, but does not introduce vicious circularity in virtue of the regularity of polymorphic terms. As recently proved ([4]), the behaviour of a polymorphic (i.e. impredicative) term on a generic imput type implies its uniform behaviour on all the input types, thus guaranteeing a generalisation based on a single generic prototype rather than the full collections of instances.

The last account I introduce in the debate and explore comes from Russell's ([10]) - long unheard - proposal to distinguish the universal propositions introduced by "all" from those introduced by "any" (and, as recently proposed, the existential propositions introduced by "some" from those introduced by "a" - cf. [12]). Substructural insights into the ambiguity of quantifiers allow us to distinguish additive from multiplicative meanings of quantification. While the universal multiplicative quantifier requires an evaluation on all the possible instances, the additive counterpart is based on the evidence of any (then a singular and generic) instance.[2] These two kinds of quantifiers inherit the properties of the corresponding multiplicative and additive conjunction and disjunction ([9], [12]). In particular, the non-contractive substructural approach allows - by renouncing the metarule of adjunction - to distinguish two forms of universal generalisation and to formalise the distinction between anything and everything. For these reasons, they seem to be particularly useful for disentangling the different meanings of quantification, on which the phenomenon of impredicativity depends, and for providing a new kind of schematic generality.

As noted above, all the approaches share the idea that the account of predicativism depends strictly on the notion of generality involved in the quantification (on which the impredicativity depends) and in the totality mentioned in the VCP. However, on the semi-intuitonistic route, we still presuppose a notion of *proper* totality, namely the kind of generality inherent in the infinite nature of every infinite collections (e.g. integers, real numbers...); on the other hand, on the type-theoretic and substructural routes, we presuppose what we can call *schematic* totality, namely the generality of the syntactic rules governing substitution, instantiation and elimination processes.

Finally, the recent thesis of predicativism as a form of potentialism ([6]) will be discussed, particularly in the light of the last two accounts of impredicativity presented above. The strategy followed so far is compatible with the potentialist thesis, but, by prioritising the choice of the logic respect to the analysis of the domain of quantification, it aims to provide an analysis that could in principle be neutral with respect to this thesis.

[2] Correspondingly, while the multiplicative existential quantification is verified on the basis of a connection between all the possible instances, the additive counterpart requires a single witness.

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