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On choice principles and their equivalents for infinite graphs

Abstract:
While it is well established that many classical graph-theoretical properties -- such as the existence of chromatic numbers or spanning trees -- are equivalent to the Axiom of Choice (AC) over Zermelo-Fraenkel set theory (ZF), the precise strength of these properties often remains unclear when restricted to specific classes of graphs. One such class is that of infinite, locally finite, connected graphs, which play an essential role in reverse mathematics. In this talk, we analyze several covering and colouring properties within this class, including the existence of minimal dominating sets, minimal edge covers, maximal matchings, as well as parameters like the chromatic number, distinguishing number and index, and many further variants. We show that these properties are, in fact, equivalent to the weak choice principle AC_{fin}^\omega over ZF. As a significant consequence, numerous Brooks'-type theorems, which provide upper bounds for these colouring invariants, fail in ZF alone, highlighting the necessity of weak choice even in highly structured, simple infinite graph classes. Finally, we touch upon the connection with Dependent Choice, as well as the status of the aforementioned colouring parameters in the framework of reverse mathematics, particularly with respect to Weak Kőnig's Lemma (WKL).