On a Second-order Generalisation of Russellian Theory of Definite Descriptions

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Definite descriptions are usually treated as first-order expressions denoting unique objects satisfying certain properties. In this talk, we intend to propose their second-order generalisation that refers to unique relations or unique properties. We investigate this subject within the paradigm of Russell's theory of definite descriptions. Second-order logic is known to be incomplete, but its fragment defined my the means of Henkin's general models is complete [1]. We formulate our theory within this complete fragment and formalize it using a cut-free sequent calculus based on Indrzejczak and Kürbis' [2] one for the first-order version of Russell's theory [4, 5].

We adhere to the exposition of Russell's theory as articulated by Indrzejczak and Zawidzki [3] as well as Indrzejczak and Kürbis [2]. Definite descriptions are defined using the λ-operator as follows:

$$(\lambda x\psi)\iota y\varphi \leftrightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$$

Consider the standard first-order language with identity. It is extended by the following expressions: a predicate abstract $(\lambda x\varphi)$, where φ is a formula; a quasi-term $\iota x\varphi$, where φ is a formula and x is an individual variable; and a formula (lambda atom) φt, where φ is a predicate abstract and t a term or quasi-term. It is a language of the logic **RL** described in [3, 2]. Let us further expand this language. We incorporate relational variables $X, Y, Z, X_1, \ldots$, second order identity $X = Y$ understood as $\forall x_1 \ldots \forall x_n\big(X(x_1, \ldots, x_n) \leftrightarrow Y(x_1, \ldots, x_n)\big)$, an atomic formula $X(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and X is an n-ary relational variable. Finally, we incorporate the subsequent expressions:

- If φ is a formula and X is a relational variable, then $\forall X\varphi$ and $\exists X\varphi$ are formulas.
- If φ is a formula, then $(\lambda X\varphi)$ is a relational abstract.
- If φ is a formula, then $\iota X\varphi$ is a pseudo-term.
- If $(\lambda X\psi)$ is a relational abstract and $\iota Y\varphi$ is a pseudo-term, then $(\lambda X\psi)\iota Y\varphi$ is a formula.

We now possess the language of the logic $\mathbf{RL^2}$, the second-order generalization of **RL**.

Consider a standard model (for the first-order logic with identity) $M = \langle D, I\rangle$ with an assignment v from the set of variables to D. *Henkin's general model* is a pair $\mathfrak{M} = \langle M, G\rangle$, where $M = \langle D, I\rangle$ is the above defined model and G is a set of subsets, relations (of any arity) on D. Let an assignment be extended for the case of relational variables. We write v^X_O to denote the X-variant of

v with $v^X_O(X) = O$, where $O \in G$. We define the notion of satisfaction of a formula φ with v in a general model, symbolically $\mathfrak{M}, v \models \varphi$, for second-order formulas as follows:

$$\mathfrak{M}, v \models X = Y \text{ iff } v(X) = v(Y),$$

$$\begin{aligned}\mathfrak{M}, v \models (\lambda X\psi)\iota Y\varphi \text{ iff there is an } O \in G \text{ such that } \mathfrak{M}, v^X_O \models \psi,\\ \mathfrak{M}, v^X_O \models \varphi^Y_X, \text{ and for any } Y\text{-variant } v' \text{ of } v^X_O,\\ \text{if } \mathfrak{M}, v' \models \varphi, \text{ then } v'(Y) = O\end{aligned}$$

$$\mathfrak{M}, v \models \forall X\varphi \text{ iff } \mathfrak{M}, v^X_O \models \varphi, \text{ for all } O \in G,$$

$$\mathfrak{M}, v \models \exists X\varphi \text{ iff } \mathfrak{M}, v^X_O \models \varphi, \text{ for some } O \in G.$$

Notice that if G contains all the relations on D, then we get the standard semantics for second-order logic which is known to be incomplete.

We extend the sequent calculus presented in [2] by the following rules for the second-order formulas, where parameters play the role of the free variables:

$$(=^2\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, X^{x_1,\ldots,x_n}_{b_1,\ldots,b_n}, Y^{x_1,\ldots,x_n}_{b_1,\ldots,b_n} \quad X^{x_1,\ldots,x_n}_{b_1,\ldots,b_n}, Y^{x_1,\ldots,x_n}_{b_1,\ldots,b_n}, \Gamma \Rightarrow \Delta}{X = Y, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=^2)\ \frac{X^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}, \Gamma \Rightarrow \Delta, Y^{x_1,\ldots,x_n}_{a_1,\ldots,a_n} \quad Y^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}, \Gamma \Rightarrow \Delta, X^{x_1,\ldots,x_n}_{a_1,\ldots,a_n}}{\Gamma \Rightarrow \Delta, X = Y}$$

$$(\exists^2 \Rightarrow)\ \frac{\varphi^X_A, \Gamma \Rightarrow \Delta}{\exists X\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \exists^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_B}{\Gamma \Rightarrow \Delta, \exists X\varphi} \qquad (\iota^2_1 \Rightarrow)\ \frac{\varphi^Y_A, \psi^X_A, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\forall^2 \Rightarrow)\ \frac{\varphi^X_B, \Gamma \Rightarrow \Delta}{\forall X\varphi, \Gamma \Rightarrow \Delta} \qquad (\iota^2_2 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \varphi^Y_C \quad B = C, \Gamma \Rightarrow \Delta}{(\lambda X\psi)\iota Y\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \forall^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^X_A}{\Gamma \Rightarrow \Delta, \forall X\varphi} \qquad (\Rightarrow \iota^2)\ \frac{\Gamma \Rightarrow \Delta, \varphi^Y_B \quad \Gamma \Rightarrow \Delta, \psi^X_B \quad \varphi^Y_A, \Gamma \Rightarrow \Delta, A = B}{\Gamma \Rightarrow \Delta, (\lambda X\psi)\iota Y\varphi}$$

where $a_1, \ldots, a_n$ are fresh individual parameters, not present in Γ and Δ; $b_1, \ldots, b_n$ are arbitrary individual parameters; A is a fresh relational parameter, not present in Γ, Δ and φ; B and C are arbitrary relational parameters.

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