Płonka sum decompositions of residuated semigroups

S. Bonzio[1], J. Gil-Férez[2], P. Jipsen[2], A. Přenosil[3], and M. Sugimoto[2]

[1] University of Cagliari, Italy
stefano.bonzio@unica.it
[2] Chapman University, USA
gilferez@chapman.edu
[3] University of Barcelona, Spain
adam.prenosil@gmail.com

The Płonka sum is a universal algebraic construction which glues together a family of algebras $\mathbf{A}_i$ indexed by a join semilattice $\mathbf{I}$, given a suitable family of connecting homomorphisms $\phi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$ for $i \leq j$ in $\mathbf{I}$. Płonka sums have successfully been used to provide decompositions of various classes of residuated structures, such as commutative idempotent involutive residuated lattices [2] and locally integral involutive pomonoids [1]. We show that a generalization of this construction allows us to decompose a much wider class of residuated structures, so-called steady residuated posemigroups, as generalized Płonka sums of simpler components.

A *residuated partially ordered semigroup*, or *residuated semigroup* for short, is a structure of the form $\mathbf{A} = \langle A, \leqslant, \cdot, \backslash, / \rangle$ such that $\langle A, \leqslant \rangle$ is a poset, $\langle A, \cdot \rangle$ is a semigroup, and the operations $\backslash$ and $/$ are the *residuals* of the *multiplication* $\cdot$, that is

$$x \cdot y \leqslant z \quad \iff \quad x \leqslant z/y \quad \iff \quad y \leqslant x\backslash z.$$

The class of residuated semigroups are only a po-variety in the sense of Pigozzi [3], i.e. a class by inequations where all fundamental operations are either order-preserving or order-reversing in each argument. An element u of a residuated semigroup $\mathbf{A}$ is called a *global identity* if $u \cdot a = a = a \cdot u$ for all $a \in A$. We call the structure $\widehat{\mathbf{A}} = \langle A, \leqslant, \cdot, 1, \backslash, / \rangle$ a *residuated partially ordered monoid*, or *residuated monoid* for short, if 1 is a global identity.

Examples of residuated semigroups and residuated monoids include all groups (ordered by the antichain order), partially ordered groups, hoops, Brouwerian semilattices and generalized Boolean algebras, as well as all subreducts of residuated lattices. In the case of groups with the antichain order, the residuals are $x\backslash y := x^{-1}y$ and $x/y := xy^{-1}$. Under this interpretation of the residuals, groups satisfy the identity $x\backslash x \approx x/x$, which can fail in relation algebras and complex algebras of groups. Residuated monoids also satisfy $y \leqslant (x/x)y$ and $y \leqslant y(x/x)$. Residuated semigroups in which these three (in)equations hold are called *balanced*.

A balanced residuated semigroup $\mathbf{A}$ can be partitioned into a family of sets indexed by the positive idempotents p of $\mathbf{A}$ (that is, $a \leqslant pa$ and $a \leqslant ap$ for all $a \in A$), namely

$$A_p := \{a \in A : a\backslash a = p\} = \{a \in A : a/a = p\}.$$

If these partition classes are universes of subalgebras of $\mathbf{A}_p$, then each $\mathbf{A}_p$ has p as its global identity. Moreover, the positive idempotents forms a join semilattice $\mathbf{E}^+\mathbf{A} := \langle \mathrm{E}^+\mathbf{A}, \cdot \rangle$, since each positive idempotent p of a balanced residuated semigroup is central (that is, $pa = ap$ for all $a \in A$). In other words, we obtain a family of residuated monoids $\mathbf{A}_p$ indexed by the join semilattice $\mathrm{E}^+\mathbf{A}$. Each such monoid has p as its only positive idempotent. This means that it is *integrally closed*: it satisfies the equations $x\backslash x \approx 1 \approx x/x$.

The Płonka sum construction allows one to reconstruct $\mathbf{A}$ from the family $\mathbf{A}_p$, given some extra data. For each $p \leqslant q$ in $\mathrm{E}^+\mathbf{A}$, we define the map $\varphi_{pq} \colon A_p \to A_q$ as $\varphi_{pq}(a) := aq$.

The multiplication operation of **A** may then be constructed as the following operation on the (disjoint) union of the algebras $\mathbf{A}_p$: for $a \in \mathbf{A}_p$ and $b \in \mathbf{A}_q$ and $r := p \vee q$ in $\mathbf{I}$

$$a \cdot b = \varphi_{pr}(a) \cdot_r \varphi_{qr}(b).$$

The maps φ_{pq} are in fact sufficient to reconstruct **A** in the special case of locally integral involutive pomonoids. However, beyond this case, further structure is needed, namely the maps $\psi_{pq}(a) := q\backslash a$ for $p \leqslant q$ in $\mathrm{E}^+\mathbf{A}$. With these maps, the residuals and the order of **A** can be reconstructed as

$$a\backslash b = \varphi_{pr}(a)\backslash_r\psi_{qr}(b), \qquad a/b = \psi_{pr}(a)/_r\varphi_{qr}(b),$$

and

$$a \leqslant b \iff \varphi_{pr}(a) \leqslant_r \psi_{qr}(a).$$

This involves generalizing the Płonka sum construction to multiple families of connecting maps.

We describe the appropriate universal algebraic generalization of Płonka sums of semilattice directed systems of homomorphisms, namely Płonka sums of semilattice directed systems of metamorphisms. We also describe the class of residuated posemigroups which admit this type of decomposition. This class subsumes earlier examples of Płonka sum decomposable residuated semigroups as special cases.

References

[1] José Gil-Férez, Peter Jipsen, and Siddhartha Lodhia. The structure of locally integral involutive po-monoids and semirings. In *Relational and algebraic methods in computer science*, volume 13896 of *Lecture Notes in Comput. Sci.*, pages 69–86. Springer, Cham, 2023.

[2] Peter Jipsen, Olim Tuyt, and Diego Valota. The structure of finite commutative idempotent involutive residuated lattices. *Algebra Universalis*, 82(4):Paper No. 57, 23, 2021.

[3] Don Pigozzi. Partially ordered varieties and quasivarieties. Technical report, Iowa State University, January 2004. https://www.researchgate.net/publication/250226836_partially_ordered_varieties_and_quasivarieties.