Comparing properties of isomorphic copies of natural numbers with primitive recursive successor

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Based on joint work with Dariusz Kalociński and Luca San Mauro.

All computable copies of $\mathcal{S} = (\mathbb{N}, Succ)$ are basically 'the same' from the perspective of the classic computable structure theory: isomorphisms between any such copies of that structure are computable and all these copies have the same class of computable functions. We use the recently emerged framework of punctual structure theory to show that the landscape is very different if we consider primitive recursive properties of copies of $\mathcal{S}$ with the domain $\mathbb{N}$ and with the primitive recursive successor. Any such copies are called punctual.

If $\mathcal{A}$ is a copy of $\mathcal{S}$, then we define classes $PR^C(\mathcal{A})$ and $PR^I(\mathcal{A})$, corresponding respectively to the set of functions whose images are p.r. in $\mathcal{A}$ and those that can be obtained from some basic set of functions using composition and primitive recursion operator. These approaches can be viewed as a semantic and a syntactic concept of a primitive recursive function. We proved that these notions converge only if $\mathcal{A}$ has the same punctual degree as $\mathcal{S}$ (as these degrees are defined in [1]). In any other case, the sets $PR^C(\mathcal{A})$ and $PR^I(\mathcal{A})$ are incomparable with respect to $\subseteq$ both with each other and with the standard class of primitive recursive functions.

We provided a characterisation of PR^C in terms of punctual degrees and of PR^I in terms of newly introduced positional degrees. We proved some differences between the properties of punctual and positional degrees.

We also showed that for every punctual copy $\mathcal{A}$ of a non-standard punctual degree there is a computable copy $\mathcal{B}$ such that $PR^C(\mathcal{A}) = PR^I(\mathcal{B})$ and $PR^I(\mathcal{A}) = PR^C(\mathcal{B})$ and that any such copy $\mathcal{B}$ cannot be punctual.

References

1. Marina Dorzhieva, Rodney Downey, Ellen Hammatt, Alexander G. Melnikov, and Keng Meng Ng. Punctually presented structures II: Comparing presentations. *Archive for Mathematical Logic*, 64(1):159–184, 2025.