On congruences of Płonka sums

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1 Introduction

The Płonka sum is a construction introduced in the 1960s in Universal Algebra by the eponymous Polish mathematician [2] (see also [4, 1]) that allows to construct a new algebra out of a semilattice direct system of similar (disjoint) algebras, called the fibers (of the system). The theory of Płonka sums has been mostly studied in the case of a similarity type without constant functional symbols: in such a case the fibers are subalgebras of their Płonka sum.
Płonka sums are strictly connected with regular identities. Recall that an identity $\alpha \approx \beta$ (in an algebraic language τ and over some set of variables X) is *regular* if $Var(\alpha) = Var(\beta)$. An identity $\alpha \approx \beta$ is valid in the Płonka sum over a non-trivial semilattice direct system $\mathbb{A} = ((\mathbf{A}_i)_{i\in I}, (I, \leq), (p_{ij})_{i\leq j})$ (i.e. $|I| \geq 2$) if and only if it is a regular identity valid in each of the fibers of $\mathbb{A}$.
Given a class of similar algebras $\mathcal{K}$, its *regularization* is the variety $\mathcal{R}(\mathcal{K})$ defined by the regular identities valid in $\mathcal{K}$. This variety is particularly interesting when the class $\mathcal{K}$ is a strongly irregular τ-variety $\mathcal{V}$ - an assumption that includes almost all examples of known irregular varieties -, i.e. a variety satisfying an identity of the form $p(x, y) \approx x$ for some binary τ-term p: in such a case, every algebra in $\mathcal{R}(\mathcal{V})$ is the Płonka sum over a semilattice direct system (with zero) of algebras in $\mathcal{V}$.

In [1] a very natural (open) **problem** is posed: to describe the congruence lattice of algebras in regular varieties.

In this talk, we will give a brief overview of the theory of Płonka sums over an algebraic language *with constant symbols* [3], with a particular emphasis on the structural aspects. Then we will address the aforementioned problem in the case of algebras in the regularization of a strongly irregular variety.

2 Congruences of Płonka Sums

Let $\mathbb{A} = ((\mathbf{A}_i)_{i\in I}, (I, \leq), (p_{ij})_{i\leq j})$ be a semilattice direct system in a strongly irregular τ-variety $\mathcal{V}$, whose strongly irregularity is witnessed by a binary term $\cdot(x, y)$, and $\mathbf{A}$ its Płonka sum. Let's begin our investigation by starting with a congruence and trying to deduce its essential structural features.

Let $\theta \in Con(\mathbf{A})$, then some very natural objects can be associated with it:

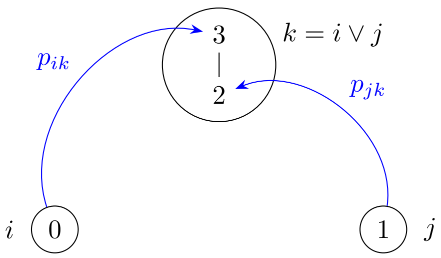
- $\forall (i, j) \in I \times I : \theta_{ij} := \theta \cap (A_i \times A_j)$;
- $S_\theta := \{(i, j) \in I \times I \mid \theta_{ij} \neq \emptyset\}$.

Observe that the strong irregularity of $\mathcal{V}$ provides the following useful fact.

Lemma 1. *Let τ be any algebraic language, then $\forall (i,j) \in S_\theta, \forall a \in A_i : (a, p_{ii\vee j}(a)) \in \theta$. Moreover, S_θ is a reflexive and symmetric subsemilattice of $\mathbf{I} \times \mathbf{I}$.*

Unfortunately, *transitivity is not guaranteed*, as the following example illustrates.

Example 1. Consider the following Płonka sum, where the two bottom fibers are trivial lattices, while the top fiber is the two-element lattice.



Then $\theta = [[0,3],[1,2]]$ is a congruence on $\mathbf{A}$ with S_θ not being transitive, since $(i,k),(k,j) \in S_\theta$, but $(i,j) \notin S_\theta$.

However, a (kind of) *weak* form of *transitivity*, outlined in the following Lemma, is always valid.

Lemma 2. *Let τ be any algebraic language, then $\forall i,j,k \in I : (i,j),(j,k) \in S_\theta \Rightarrow (i, i\vee k) \in S_\theta$.*

To simplify the exposition, we will say that S_θ is **upper transitive**.
Actually, thanks to Lemmas 1 and 2, we can provide an exact characterization for transitivity.

Lemma 3. *Let τ be any algebraic language. For every $i,j,k \in I$ such that $(i,j),(j,k) \in I$ T.F.A.E.*

1. $(i,k) \in S_\theta$;
2. $(p_{ii\vee k} \times p_{ki\vee k})^{-1}(\theta_{i\vee k, i\vee k}) \neq \emptyset$.

Remark 1. Observe that in Example 1 we have $(p_{ik} \times p_{jk})^{-1}(\theta_{kk}) = \emptyset$, and this explains the non-transitivity of S_θ.

In some particular, yet relevant, cases, S_θ turns out to be a congruence on $\mathbf{I}$.

Corollary 1. *Let τ be any algebraic language. If one of the following occurs:*

(i) $\mathbf{I}$ *is a chain;*

(ii) τ *be an algebraic language containing constants*

then $S_\theta \in Con(\mathbf{I})$.

Consequently, transitivity is always ensured for algebraic languages having constants.
The necessary conditions that we have highlighted through the preceding Lemmas provide a complete characterization of an element of $Con(\mathbf{A})$, as illustrated by the following Theorem.

Theorem 1. *Let τ be any algebraic language. Let $S \subseteq I \times I$ and $(\theta_{ii})_{i\in I}$ be a family such that the following conditions occur:*

(i) S *is a reflexive, symmetric and upper transitive subsemilattice of* $\mathbf{I} \times \mathbf{I}$;

(ii) $\forall i \in I : \theta_{ii} \in Con(\mathbf{A}_i)$;

(iii) $\forall (i,j) \in I \times I : \theta_{ii} \subseteq (p_{ii\vee j} \times p_{ii\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *with equality if* $(i,j) \in S$;

(iv) $\forall (i,j) \in I \times I : (i,j) \in S \iff (i, i\vee j), (j, i\vee j) \in S, (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j}) \neq \emptyset$

For every $(i,j) \in S \setminus \Delta_{\mathbf{I}}$, *let* $\theta_{ij} := (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *then*

$$\theta := \bigcup_{(i,j)\in S} \theta_{ij} \in Con(\mathbf{A}).$$

Furthermore, all the elements of $Con(\mathbf{A})$ *arise in this way.*

In the case of an algebraic language containing constants, the characterization takes on a simpler form.

Theorem 2. *Let* τ *be an algebraic language containing constants (or suppose* $\mathcal{V}$ *admits an algebraic constant). Let* $S \subseteq I \times I$ *and* $(\theta_{ii})_{i\in I}$ *a family such that the following conditions occur:*

(i) $S \in Con(\mathbf{I})$;

(ii) $\forall i \in I : \theta_{ii} \in Con(\mathbf{A}_i)$;

(iii) $\forall (i,j) \in I \times I : \theta_{ii} \subseteq (p_{ii\vee j} \times p_{ii\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *with equality if* $(i,j) \in S$;

(iv) $\forall (i,j) \in S : (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j}) \neq \emptyset$.

For every $(i,j) \in S \setminus \Delta_{\mathbf{I}}$ *let* $\theta_{ij} := (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *then*

$$\theta := \bigcup_{(i,j)\in S} \theta_{ij} \in Con(\mathbf{A}).$$

Furthermore, all the elements of $Con(\mathbf{A})$ *arise in this way.*

References

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