Canonical formulas for substructural logics

Nick Galatos
(Joint work with Kempton Albee)

University of Denver

CLoCk, Krakow, June 2025

Outline

- Canonical formulas
- Substructural Logics and Residuated lattices
- The substructural hierarchy
- Sufficient condition via 3 properties
- Conic idempotent n-cyclic
- Weakly commutative and (n, m)-potent

Goal

Using a generalization of Jankov formulas [Jankov 1968], [Chagrov-Zakharyaschev] axiomatized all varieties of Intuitionistic logic/Heyting algebras in a unifrom way by *canonical* formulas.

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4. Canonical formulas are in the $\mathcal{N}_4$ level of the substructural formula hierarchy and they all have a uniform shape. So for varieties admitting canonical formulas, the substructural hierarchy stabilizes.

Substructural logics

Classical logic studies truth.

Intuitionistic logic (Brouwer, Heyting) deals with provability or constructibility.
The algebraic models are Heyting algebras.

Many-valued logic (Łukasiewicz) allows different degrees of truth. [Ulam's game]
$[x \to (x \to y)] \to (x \to y)$ is not a theorem. The algebraic models fail $x \leqslant x \cdot x$.

Relevance logic (Anderson, Belnap) deals with relevance.
$p \to (q \to q)$ is not a theorem. The algebraic models do not satisfy integrality $x \leqslant 1$.
$p \to (\neg p \to q)$ [or $(p \cdot \neg p) \to q$] is not a theorem, where $\neg p = p \to 0$. The algebraic models do not satisfy $0 \leqslant x$.

Linear logic (Girard) studies preservation of resourses.
$p \to (p \to p)$ [or $(p \cdot p) \to p$] and $p \to (p \cdot p)$ are not theorems.
The algebraic models do not satisfy mingle $x^2 \leqslant x$ nor contraction $x \leqslant x^2$.

The calculi for substructural logics have variants used in:

- Mathematical linguistics: Context-free grammars, pregroups. (Lambek, Buzskowski)
- CS: Memory allocation, pointer management, concurrent programming. (Separation logic, bunched implication logic).

Gentzen's system **LJ** for intuitionistic logic

A ***sequent*** is an expression $a_1, \ldots, a_n \Rightarrow a_0$, where a's are formulas. For $a, b, c \in Fm$, $x, y, z, x_1, x_2 \in Fm^*$, we have the inference rules:

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ \text{(cut)} \qquad \frac{}{a \Rightarrow a}\ \text{(Id)}$$

$$\frac{y, x_1, x_2, z \Rightarrow c}{y, x_2, x_1, z \Rightarrow c}\ \text{(e)} \qquad \frac{y, z \Rightarrow c}{y, x, z \Rightarrow c}\ \text{(w)} \qquad \frac{y, x, x, z \Rightarrow c}{y, x, z \Rightarrow c}\ \text{(c)}$$

$$\frac{y, a, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathrm{L}\ell) \qquad \frac{y, b, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathrm{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathrm{R})$$

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \vee b, z \Rightarrow c}\ (\vee \mathrm{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}r)$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, a \to b, z \Rightarrow c}\ (\to \mathrm{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \to b}\ (\to \mathrm{R})$$

$$\frac{y, z \Rightarrow c}{y, 1, z \Rightarrow c}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

Basic substructural logics

In $\mathbf{LJ}$, the sequent $a_1, \ldots, a_n \Rightarrow a_0$ is provable iff the sequent $a_1 \wedge \ldots \wedge a_n \Rightarrow a_0$ is, so comma corresponds to $\wedge$. The proof system $\mathbf{FL}$ of Full Lambek calculus is obtained from Gentzen's proof system $\mathbf{LJ}$ for intuitionistic logic by removing the three basic structural rules:

$\dfrac{u[x,y] \Rightarrow c}{u[y,x] \Rightarrow c}\ (e)$	(exchange)	$[x \to (y \to z)] \to [y \to (x \to z)]$	$xy \leqslant yx$
$\dfrac{u[x,x] \Rightarrow c}{u[x] \Rightarrow c}\ (c)$	(contraction)	$[x \to (x \to y)] \to (x \to y)$	$x \leqslant x^2$
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$$\frac{u[\varepsilon] \Rightarrow c}{u[x] \Rightarrow c}\ (i) \qquad \text{(integrality)} \qquad y \to (x \to y) \qquad x \leqslant 1$$

In $\mathbf{FL}$, comma and $\wedge$ do not correspond any more. But we can conservatively add a new connective $\cdot$ (*fusion* or *multiplication*) that does correspond to comma and rules:

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot \mathrm{R})$$

Also, $a \to b$ splits into $a \backslash b$ and b / a.

FL

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$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c} \ (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} \ (\cdot \mathsf{R})$$

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where $a, b, c \in Fm$, $x, y, z \in Fm^*$. Extensions of $\mathbf{FL}$ are known as *substructural logics*.

Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
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Examples:

1. Boolean and Heyting algebras, where $x \cdot y = x \wedge y$ and $x \rightarrow y = x\backslash y = y/x$.
   We also add a constant 0 and define $\neg x = x \rightarrow 0$.
2. Also, MV-algebras and other algebras of substructural logics:
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3. Lattice-ordered groups: $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. (and ℓ-pregroups)
4. Quantales (relating to quantal-valued model theory, C*-algebras)
5. Relation algebras: $R\backslash S = (R^{\cup} \circ S^{c})^{c}$, $S/R = (S^{c} \circ R^{\cup})^{c}$.
6. Lattices of ideals of rings, under the usual multiplication and division of ideals. (Ward and Dilworth 1930's)
7. Computer Science: Action algebras, Kleene algebras with tests. (Pratt, Kozen)

Varieties of residuated lattices form *equivalent algebraic semantics* (a la Lindenbaum-Blok-Pigozzi) for various substructural logics.

Beyond sequent rules

By results of [G.-Jipsen, TAMS, 2013], $\{\vee, \cdot, 1\}$-equations give rise to analytic structural *sequent* rules (cut elimination holds).

By results of [Ciabbatoni-G.-Terui, LICS, 2008] and [G.-Ciabbatoni-Terui, APAL, 2012] strongly analytic sequent rules are essentially defined only by $\{\vee, \cdot, 1\}$-equations.

A hypersequent is a multiset $s_1 \mid \cdots \mid s_m$ of sequents s_i. Hypersequent structural rules:

$$\frac{H \mid s_1' \quad H \mid s_2' \quad \ldots \quad H \mid s_n'}{H \mid s_1 \mid \cdots \mid s_m}$$

Hypersequent calculi allow for the proof-theoretic study of many more extensions, such as the Gödel-Dummet logic modeled by $(x \to y) \vee (y \to x)$, as $\mid$ is a form of disjunction.

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1. The proof theory of (sequent and) hypersequent calculi (cut elimination), including proceedures for obtaining analytic rules from axioms
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3. connections to (positive universal) classes of FSIs of the corresponding variety
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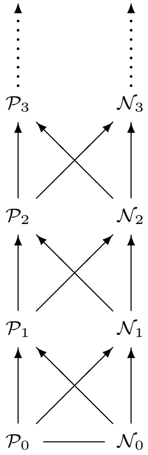
A blueprint for more expressive axioms is given by the *substructural hierarchy* (similar to the arithmetical hierarchy) is defined by alternations of *positive* and *negative* connectives.

Substructural hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:
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  - (P2) $a, b \in \mathcal{P}_{n+1} \;\Rightarrow\; a \vee b, a \cdot b, 1 \in \mathcal{P}_{n+1}$
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  - (N3) $a \in \mathcal{P}_{n+1}, b \in \mathcal{N}_{n+1} \;\Rightarrow\; a\backslash b, b/a, 0 \in \mathcal{N}_{n+1}$
- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\wedge, \mathcal{P}_{n+1}\backslash, /\mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$
  $p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leqslant r$
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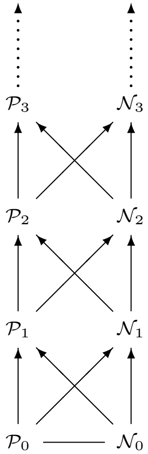

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Partial attempts to handle the $\mathcal{N}_4$ level include:
[G.-Metcalfe, APAL 2016] proof theory and complexity (coNP-complete) for ℓ-groups.
[Colacito-G.-Metcalfe, Santchi JoA 2022] decidability for distributive ℓ-monoids.

The formulas

Given a finite algebra $\mathbf{A}$ and $D^{\wedge}, D^{\backslash}, D^{/} \subseteq A^2$, for each $a \in A$, we introduce a new variable X_a, and we set:

$$\begin{aligned}\Gamma := & (X_{\bot} \leftrightarrow \bot) \wedge (X_1 \leftrightarrow 1) \wedge \\ & \bigwedge \{X_{a \cdot b} \leftrightarrow X_a \cdot X_b \mid a, b \in A\} \wedge \\ & \bigwedge \{X_{a \vee b} \leftrightarrow X_a \vee X_b \mid a, b \in A\} \wedge \\ & \bigwedge \{X_{a \wedge b} \leftrightarrow X_a \wedge X_b \mid (a, b) \in D^{\wedge}\} \wedge \\ & \bigwedge \{X_{a \backslash b} \leftrightarrow X_a \backslash X_b \mid (a, b) \in D^{\backslash}\} \wedge \\ & \bigwedge \{X_{a / b} \leftrightarrow X_a / X_b \mid (a, b) \in D^{/}\}\end{aligned}$$

and

$$\Delta := \bigvee \{X_a \backslash X_b \wedge 1 \mid a, b \in A \text{ with } a \nleq b\}.$$

For brevity we set $D := (D^{\wedge}, D^{\backslash}, D^{/})$. The $\{\vee, \cdot, 1\}$-*canonical formula* $\delta_\tau(\mathbf{A}, D)$ associated with $\mathbf{A}$, D, and a unary term τ is defined as follows: (and is in $\mathcal{N}_4$)

$$\delta_\tau(\mathbf{A}, D) := \tau(\Gamma) \backslash \Delta.$$

The 3 properties: necessary condition

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Given a unary term τ, we say that a variety of residuated lattices has the *τ-deduction theorem* if all of its algebras $\mathbf{A}$ do (for the same τ): for all $x, y \in A$ and $X \subseteq A$,

$$y \in F(X \cup \{x\}) \text{ iff } \tau(x)\backslash y \in F(X).$$

Theorem. A variety of residuated lattices has the τ-deduction theorem iff the corresponding substructural logic does. (EDPC follows.)

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Given a unary term τ, we say that a variety of residuated lattices has the *τ-deduction theorem* if all of its algebras $\mathbf{A}$ do (for the same τ): for all $x, y \in A$ and $X \subseteq A$,

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Theorem. A variety of residuated lattices has the τ-deduction theorem iff the corresponding substructural logic does. (EDPC follows.)

An isomorphism class $\mathcal{K}$ of residuated lattices has the *$\{\vee, \cdot, 1\}$-FEP* if for every $\mathbf{A} \in \mathcal{K}$ and finite subset X of A, there exists a finite algebra $\mathbf{B} \in \mathcal{K}$ that is a $\{\cdot, \vee, 1\}$-subalgebra of $\mathbf{A}$, it contains X and $x, y, x \bullet^{\mathbf{A}} y \in X$ implies $x \bullet^{\mathbf{B}} y = x \bullet^{\mathbf{A}} y$, for $\bullet \in \{\wedge, \backslash, /\}$.

The 3 properties: necessary condition

A class of residuated lattices has *SI-opremums* if every subdirectly irreducible algebra in the class has an opremum: an element $s < 1$ where $x < 1$ implies $x \leqslant s$, for all $x \in A$.

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We say that $\mathcal{K}$ has the *$\{\vee, \cdot, 1\}$-bFEP* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $\mathbf{A} \in \mathcal{K}$ and every finite subset X of A, there exists a $\mathbf{B}$ that witnesses the $\{\vee, \cdot, 1\}$-FEP for $\mathbf{A}$ and X, and $|B| \leqslant f(|X|)$.

Proof idea (for a variety $\mathcal{V}$ satisfying the 3 conditions)

Let $\mathcal{V} \not\models \varphi$, let $\mathsf{Sub}(\varphi)$ be the collection of all subformulas of φ, and let $(\mathbf{A}_1, v_1), \ldots, (\mathbf{A}_m, v_m)$ be all pairs such that the $\mathbf{A}_i$'s are, up to isomorphism, all algebras in $\mathcal{V}_{\mathsf{SI}}$ with size up to $f(|\mathsf{Sub}(\varphi)|)$, and v_i is a valuation such that $(\mathbf{A}_i, v_i) \not\models \varphi$.

$$D_i^{\wedge} := \{(a,b) \in (\mathsf{Sub}_{v_i}(\varphi))^2 \mid a \wedge b \in \mathsf{Sub}_{v_i}(\varphi)\}$$

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$\Sigma_\phi := \{(\mathbf{A}_i, D_i^{\wedge}, D_i^{\backslash}, D_i^{/}) \mid 1 \leqslant i \leqslant m\}$ is the *system associated with* φ. (Σ_ϕ is finite.)

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Main Theorem. For every formula φ that fails in $\mathcal{V}$ and for every $\mathbf{B} \in \mathcal{V}$:

$$\mathbf{B} \models \varphi \quad \text{if and only if} \quad \mathbf{B} \models \bigwedge\{\delta_\tau(\mathbf{A}, D) \mid (\mathbf{A}, D) \in \Sigma_\phi\}$$

Via contraposition (through a lemma). [1st: no τ-DT needed. 2nd: no $\{\vee, \cdot, 1\}$-bFEP.]

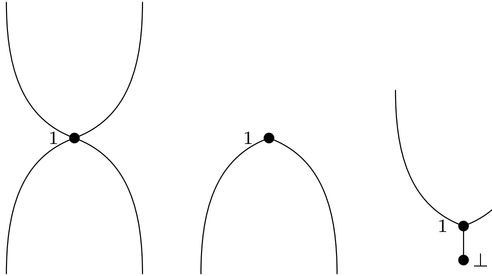
$$\mathbf{B} \not\models \varphi \iff \exists(\mathbf{A}, D) \in \Sigma_\phi,\ \exists \mathbf{C} \in \mathcal{V}_{\mathsf{SI}} : \mathbf{A} \overset{D}{\rightarrowtail} \mathbf{C} \twoheadleftarrow \mathbf{B} \iff \exists(\mathbf{A}, D) \in \Sigma_\phi,\ \mathbf{B} \not\models \delta_\tau(\mathbf{A}, D).$$

Given $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and $D^{\wedge}, D^{\backslash}, D^{/} \subseteq A^2$, a D-embedding [notation: $h : \mathbf{A} \overset{D}{\rightarrowtail} \mathbf{B}$] is a map $h : \mathbf{A} \to \mathbf{B}$, where $D := (D^{\wedge}, D^{\backslash}, D^{/})$, that is injective, preserves $\cdot$ and $\vee$, $(a,b) \in D^{\wedge}$ implies $h(a \wedge b) = h(a) \wedge h(b)$, $(a,b) \in D^{\backslash}$ implies $h(a \backslash b) = h(a) \backslash h(b)$.

Corollary. Every subvariety of $\mathcal{V}$ of is axiomatizable by $\{\vee, \cdot, 1\}$-canonical formulas.

Semiconic idempotent

A residuated lattice $\mathbf{A}$ is called *conic* if each of its elements is comparable to 1. *Integral* ($x \leqslant 1$) residuated lattices and residuated chains (i.e., totally ordered residuated lattices) are examples of conic residuated lattices.



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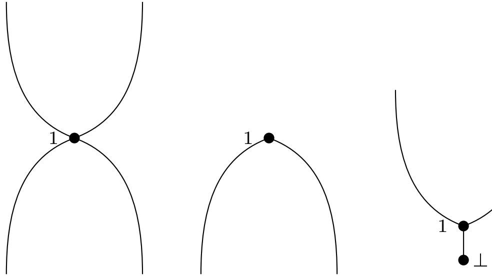

ConIdRL denotes the class of conic idempotent ($x^2 = x$) residuated lattices; subdirect products give the variety S := V(ConIdRL) of *semiconic residuated lattices*. The corresponding logic is denoted by $\mathbf{sCI}$; it includes IPC, semilinear logic and and relevance logic with mingle.

The conic idempotent residuated lattices that are integral are precisely the Heyting/Brouwerian algebras: (bounded) residuated lattices satisfying $xy = x \wedge y$.

SI-opremum and local deduction theorem

Theorem. [G.-Fussner APAL 2024] A semiconic idempotent residuated lattice has an opremum $\mathbf{A}$ iff it is subdirectly irreducible. So, the variety S has SI-opremums.

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We define the *inverses* of x as $x^{\ell} := 1/x$ and $x^{r} := x\backslash 1$. Also, we define:

$$s_1(y) := y \wedge y^{\ell\ell} \wedge y^{rr}, \qquad s_2(y) := y \wedge y^{\ell\ell\ell\ell} \wedge y^{\ell\ell rr} \wedge y^{rr\ell\ell} \wedge y^{rrrr}.$$

and, for all n,

$$s_n(y) := y \wedge \bigwedge\{y^{c_1 c_1 c_2 c_2 \cdots c_n c_n} \mid c_1, c_2, \ldots, c_n \in \{\ell, r\},\}$$

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Note that if an element is *central* ($ax = xa$, for all x) then it is *cyclic* ($a^\ell = a^r$).
$\mathsf{S}_n = \mathsf{S} + (s^{n+1}(x) = s^n(x))$ n-cyclicity. $\mathsf{S}_n^- = \mathsf{S} + (t^{n+1}(x) = t^n(x))$ negative n-cyclicity.

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Corollary. For every n, S_n^- has the t_n-deduction theorem. (Also S_n.)

Decomposition for conic idempotent residuated lattices

A *decomposition system* is a structure $(\mathbf{S}, \{\mathbf{A}_s : s \in S\})$, where $\mathbf{S}$ (called the *skeleton*) is an idempotent residuated chain, the A_s's are disjoint, and, for every $s \in S$, $\mathbf{A}_s$ (called a *component*) is a *prelattice* with top element s such that:

1. If s has no lower cover in $\mathbf{S}$, then $\mathbf{A}_s$ is a lattice.
2. For negative $s \in S$, the component $\mathbf{A}_s$ is a Brouwerian lattice; we denote by $\to_s$ its implication.
3. If s is not central, then $|A_s| = 1$.

Given a decomposition system $D = (\mathbf{S}, \{\mathbf{A}_s : s \in S\})$, we consider the ordinal sum $\bigoplus_{s\in S} \mathbf{A}_s$ and for $x \in A_s$ and $y \in A_t$, we define a residuated lattice $\mathbf{A}_D$ on it:

$$xy = \begin{cases} x \wedge y, & s = t \leqslant 1 \\ x \vee y, & s = t > 1 \\ x, & st = s \text{ and } s \neq t \\ y, & st = t \text{ and } s \neq t \end{cases} \qquad y/x = \begin{cases} s^{\ell} \vee y, & x \leqslant y \\ s^{\ell} \wedge y, & t < s, \text{ or } 1 < s = t \text{ and } x \nleqslant y \\ x \to_s y, & s = t \leqslant 1 \text{ and } x \nleqslant y \end{cases}$$

Theorem. [G.-Fussner] Given a decomposition system D, the algebra $\mathbf{A}_D$ is a conic idempotent residuated lattice. Conversely, every conic idempotent residuated lattice is of this form, where $\mathbf{S}$ is the subalgebra of $\mathbf{A}$ based on the set $\gamma[A]$, $A_s = \gamma^{-1}(s)$, for all $s \in S$, and $\gamma(x) = x^{\ell r} \wedge x^{r\ell}$.

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FEP for S

Lemma. If $\mathbf{A} \in$ ConIdRL, then the $\{\vee, \cdot, 1\}$-subalgebra of $\mathbf{A}$ generated by a given finite subset X of A has at most $2^{2^{|X|}}$ elements.

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Corollary. The variety S has the $\{\cdot, \vee, 1\}$-bFEP.

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Lemma. Given a variety $\mathcal{V}$ of residuated lattices, if $\mathcal{V}_{\mathsf{SI}}$ has the $\{\vee, \cdot, 1\}$-bFEP in $\mathcal{V}$, then $\mathcal{V}$ has the $\{\vee, \cdot, 1\}$-bFEP.

Corollary. The variety S has the $\{\cdot, \vee, 1\}$-bFEP.

Lemma. If $\mathcal{V}$ is a variety of residuated lattices that has the $\{\vee, \cdot, 1\}$-bFEP and SI-opremums, then $\mathcal{V}_{\mathsf{SI}}^+$ ($= \mathcal{V}_{\mathsf{SI}}$ plus the trivial algebra) has the $\{\vee, \cdot, 1\}$-bFEP.

FEP for S

Lemma. If $\mathbf{A} \in$ ConIdRL, then the $\{\vee, \cdot, 1\}$-subalgebra of $\mathbf{A}$ generated by a given finite subset X of A has at most $2^{2^{|X|}}$ elements.

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To establish the $\{\vee, \cdot, 1\}$-bFEP for S_n^-, we need more theory.

Flow diagrams

Let a be a positive and b a negative element of an idempotent residuated chain $\mathbf{A}$.

$a \; L \; b$ means that $\{a, b\}$ forms a left-zero semigroup: $ab = a$ and $ba = b$.
$a \; R \; b$ means that $\{a, b\}$ forms a right-zero semigroup: $ab = b$ and $ba = a$.

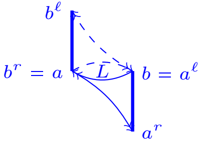
Theorem. [G.-Fussner APAL 2025] Let $\mathbf{A}$ be an idempotent residuated chain.

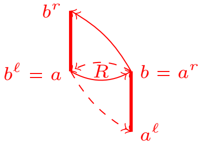
If a is a positive non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

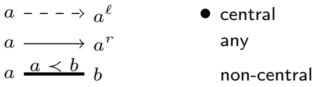
1. $a^{\ell\ell} \prec a^{\ell r} = a \; L \; a^{\ell} \succ a^{r}$.
2. $a^{rr} \prec a^{r\ell} = a \; R \; a^{r} \succ a^{\ell}$.

If b is a negative non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

1. $b^{\ell} \prec b^{r} \; L \; b = b^{r\ell} \succ b^{rr}$.
2. $b^{r} \prec b^{\ell} \; R \; b = b^{\ell r} \succ b^{\ell\ell}$.







FEP for S_n^-

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Lemma. Let $\mathbf{S}$ be an idempotent residuated chain, $x \in S$ and $n \in \mathbb{N}$.

1. x is central iff $s(x) = x$. If x is noncentral, then $s(x) \prec x$.
2. $s^n(x)$ is central iff there exists a central element $y \in S$ such that $(y, x]$ has size at most n and consists entirely of noncentral elements; in this case $y = s^n(x)$.

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Theorem. The varieties S_n and S_n^- have the $\{\vee, \cdot, 1\}$-bFEP.

Idea. We modify the construction of the algebra $\mathbf{B}$ by augmenting to the set $X_{\gamma,\sigma}$ with

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For an element x in an idempotent residuated chain, we define

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Corollary. The varieties S_n and S_n^- admit canonical formulas.

Weakly commutative and potent

We generalize the result of [Bezhanishvili-G.-Spada 2017] for residuated lattices that are commutative ($xy = yx$), integral ($x \leqslant 1$), and n-potent ($x^{n+1} = x^n$).

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Given a positive integer s and a non-constant partition $a = (a_0, a_1, \ldots, a_s) \in \mathbb{N}^{s+1}$ of $s + 1$ (i.e., not all a_i's are equal to 1 and $a_0 + a_1 + \ldots + a_s = s + 1$), we define

$$(a) \qquad xy_1xy_2 \cdots y_sx = x^{a_0}y_1x^{a_1}y_2 \cdots y_sx^{a_s}.$$

For example, $(2, 0)$ is the equation $xyx = x^2y$ and $(0, 2, 1)$ is $xyxzx = yx^2zx$.

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The congruence filter generated by a subset X of $\mathbf{A}$, denoted by $F(X)$, is the upward closure of all products of iterated conjugates of elements of X.

τ-deduction

A variety is called *s-subcommutative* if it satisfies $(x \wedge 1)^s y = y(x \wedge 1)^s$; it is called *subcommutative* if it is s-subcommutative for some $s \in \mathbb{Z}^+$.

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for some $s \in \mathbb{Z}^+$; i.e., the last coordinate a_s of the vector $\vec{a}$ is zero. Likewise, a *final* weak commutativity equation is one of the form (the first coordinate a_0 of a is zero)

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Lemma. Every subcommutative negatively k-potent variety of residuated lattices has the τ-deduction theorem, for $\tau(\varphi) = (\varphi \wedge 1)^k$.

SI-opremum

Lemma. If a and b are negative elements of a subcommutative residuated lattice, then: $a \in F(b)$ iff there exists $n_b \in \mathbb{N}$ with $b^{n_b} \leqslant a$.

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$$(a \vee b)^{2t} = \bigvee \{c_{i_1} \cdot \ldots \cdot c_{i_{2t}} \mid i_1, \ldots, i_{2t} \in \mathbf{N}, c_{i_1}, \ldots, c_{i_{2t}} \in \{a, b\}\}$$

Each $c_{i_1} \cdot \ldots \cdot c_{i_{2t}}$ contains at least t-many a's or t-many b's, so by monotonicity and integrality,

$$c_{i_1} \cdot \ldots \cdot c_{i_{2t}} \leqslant a^t \qquad \text{or} \qquad c_{i_1} \cdot \ldots \cdot c_{i_{2t}} \leqslant b^t,$$

thus $(a \vee b)^{2t} \leqslant a^t \vee b^t \leqslant a^{n_a} \vee b^{n_b} \leqslant z < 1$. Therefore, $a \vee b < 1$: 1 is join irreducible.

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If D is a chain of strictly negative elements, then for $\Phi = D^n$ (choice functions):

$$\left(\bigvee D\right)^k = \bigvee \{\varphi(1) \cdot \ldots \cdot \varphi(k) \mid \varphi \in \Phi\} = \bigvee \{(\varphi(1) \vee \cdots \vee \varphi(k))^k \mid \varphi \in \Phi\} \leqslant z$$

FEP

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$\mathcal{K}(a)$ denotes the variety of monoids axiomatized by (a).

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$\mathcal{K}(a)$ denotes the variety of monoids axiomatized by (a). For $p, q, \ell \in \mathbb{N}$ with $p + q < \ell$, $\mathcal{K}(p, q, \ell)$ denotes the variety of monoids axiomatized by the system of equations

$$w_1 w w_2 = w_1 w' w_2$$

where $\mathrm{del}_x(w) = \mathrm{del}_x(w') = y_p y_{p+1} \cdots y_{\ell-q}$, $|w|_x = |w'|_x = \ell - p - q$, $w_1 = w_{1,p}$ and $w_2 = w_{\ell-q+1,\ell}$.

Theorem. [G.-Cardona IJAC 2015] $\mathcal{K}(a)$ is a subvariety of $\mathcal{K}(p_a, q_a, 2s)$ for all (a), where $p_a := \max\{j \mid \forall i < j,\ a_i = 1\}$ and $q_a := \max\{j \mid \forall i > s - j,\ a_i = 1\}$.

FEP

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Theorem. Any variety axiomatized by (n, m)-potency, a weak commutativity equation, and a (possibly empty) set of $\{\vee, \cdot, 1\}$-equations has the $\{\vee, \cdot, 1\}$-bFEP.

Corollary. Any variety axiomatized by (n, m)-potency, an initial and a final weak commutativity equation, and any set of $\{\vee, \cdot, 1\}$-equations has canonical formulas.