Non-reducible infinite theories of the first order

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Naming parts and definitions (1)

- Theory T - **a set** of formulas.
- We consider **first order formulas and theories only!**
- Theory T is **consistent** if it is **NOT** the case that:

$$\phi \in T \textit{ and } (\neg\phi) \in T.$$

- We denote a **consequence relation** by $\vdash$.
- Structure $\mathcal{M}$ is a **model** of theory T if:

$$\forall \phi \in T : \mathcal{M} \models \phi.$$

Naming parts and definitions (2)

- If a theory is **consistent** then it has a model.
- Two theories are **equivalent** if they have the same collection of models.
- We say that theory T is **infinite** if $|T| \geqslant \aleph_0$.
- Let T_1, T_2 be theories such that $|T_2| < |T_1|$. If T_1 and T_2 are equivalent then we say that T_1 **is reducible to** T_2.

Original theorem

Theorem (Basic)

Let T be an infinite, consistent theory. If:

1. *T has no finite model,*
2. *Every finite subset of T has a finite model,*

then T is not equivalent to any finite theory.

Let us analyze the proof ...

Proof of the original theorem (1)

Let's conduct the proof **indirectly**:

- Assume T is equivalent with some finite T_0.
- Hence $\forall\phi \in T_0 : T \vdash \phi$.
- In particular, $\forall\phi \in T_0$ there exists $S_\phi \subseteq_{fin} T$ such that:

$$S_\phi \vdash \phi.$$

- Let us consider $\widehat{S} = \bigcup_{\phi \in T_0} S_\phi$.
- Since T_0 is finite and each of S_ϕ is finite the $\widehat{S}$ is finite as well.

Proof of the original theorem (2)

- Let us notice that $\widehat{S} \vdash T_0$ but also $T_0 \vdash T$ so in particular:

$$\widehat{S} \vdash T$$

- Since also $\widehat{S} \subseteq T$ then:

$$T \vdash \widehat{S}.$$

- Hence $\widehat{S}$ and T are **equivalent**.
- So $\widehat{S}$ and T **have the same models**.

Proof of the original theorem (3)

Theorem (Basic)

Let T be an infinite, consistent theory. If:

1. *T has no finite model,*
2. *Every finite subset of T has a finite model,*

then T is not equivalent to any finite theory.

- From 2. $\implies$ there is a **finite** model $\mathcal{M}_{\widehat{S}}$ of theory $\widehat{S}$.

Proof of the original theorem (4)

Theorem (Basic)

Let T be an infinite, consistent theory. If:

1. *T has no finite model,*
2. *Every finite subset of T has a finite model,*

then T is not equivalent to any finite theory.

- From 2. $\implies$ there is a **finite** model $\mathcal{M}_{\widehat{S}}$ of $\widehat{S}$.
- From 1. $\implies$ $\mathcal{M}_{\widehat{S}}$ is **not** a model of T because of $\mathcal{M}_{\widehat{S}}$ **finiteness**.
- We arrived at a contradiction.

Generalization

The previous theorem can be generalized to larger cardinalities of the set:

Theorem (Extended)

Let T be a consistent theory of infinite cardinality κ. If every proper subset $S \subseteq T$, such that $|S| < |T|$ has a model M_S which is not a model of T then T is not equivalent with any other theory $\widehat{T}$ such that $|\widehat{T}| < \kappa$.

Let us slightly modify the proof of the previous theorem ...

Proof of the modified theorem (1)

- We proceed indirectly (as in case of **Basic Theorem**)
- Assume T is equivalent with some theory $\widehat{T}$, such that:

$$|\widehat{T}| = \lambda < \kappa$$

- Hence, every formula $\phi \in \widehat{T}$ is a consequence of T, i. e. for every ϕ in $\widehat{T}$:

$$T \vdash \phi.$$

- In particular, for every $\phi \in \widehat{T}$ there exists $S_\phi \subseteq_{fin} T$ such that:

$$S_\phi \vdash \phi.$$

Proof of the modified theorem (2)

- Let:

$$\widehat{S} = \bigcup_{\phi \in \widehat{T}} S_\phi.$$

- Then:

$$\widehat{S} \vdash \widehat{T}.$$

- Also:

$$\widehat{T} \vdash T.$$

- Hence:

$$\widehat{S} \vdash T.$$

- Since:

$$\widehat{S} \subseteq T$$

  then:

$$T \vdash \widehat{S}.$$

- So T and $\widehat{S}$ are equivalent and have the same models.

Proof of the modified theorem (3)

- Let us establish the cardinality of $\widehat{S}$.
- Each of S_ϕ is finite so let us denote $|S_\phi| = n_\phi$.
- Then:

$$|\widehat{S}| = |\bigcup_{\phi \in \widehat{T}} S_\phi| = \lambda \cdot n_\phi = \lambda.$$

- But $|\widehat{S}| < |T|$ and $\widehat{S} \subseteq T$.
- Then, from the assumptions of the theorem, there is a structure $\mathcal{M}$ such that it is a model of $\widehat{S}$ but it is not a model of T.
- We arrived at a contradiction with the fact that $\widehat{S}$ and T have the same models.

Important remark

Theorem (Extended)

Let T be a consistent theory of infinite cardinality κ. If every proper subset $S \subseteq T$, such that $|S| < |T|$ has a model M_S which is not a model of T then T is not equivalent with any other theory $\widehat{T}$ such that $|\widehat{T}| < \kappa$.

Remark

The assumptions of the ***Extended Theorem*** *do not exclude the existence of structures which are models for both T and $\widehat{T}$.*

Example (1)

- Let T be a theory with only one binary relation symbol R.
- T models relation R the way it is an equivalence relation such that:
  - R has countably infinite many equivalence classes.
  - Each of these classes is itself countably infinite.

Example (1) continued

- $\forall a : aRa$,
- $\forall a \forall b \forall c : aRb \wedge bRc \implies aRc$,
- $\forall a \forall b : aRb \implies bRa$,
- $\phi_n = \forall a_1 \forall a_2 \ldots \forall a_n \exists b$:

$$\bigwedge_{1 \leqslant i,j \leqslant n, i \neq j} \neg(a_i = a_j) \wedge \bigwedge_{1 \leqslant i \leqslant n-1} (a_i R a_{i+1}) \implies (bRa_1) \wedge \bigwedge_{1 \leqslant i \leqslant n} \neg(b = a_i),$$

  for $n \in \mathbb{N}$,

- $\psi_n = \forall a_1 \forall a_2 \ldots \forall a_n \exists b$:

$$\bigwedge_{1 \leqslant i,j \leqslant n, i \neq j} \neg(a_i = a_j) \wedge \bigwedge_{1 \leqslant i \leqslant n-1} (a_i R a_{i+1}) \implies \neg(bRa_1) \wedge \bigwedge_{1 \leqslant i \leqslant n} \neg(b = a_i),$$

  for $n \in \mathbb{N}$.

Example (1) continued

- The formula ϕ_i states that given i elements in an equivalence class then there is a $(i+1)^{th}$ element in this class as well.
- The formula ψ_i states that given i elements in an equivalence class then there is another element which does not belong to this class.
- Let us assume that $T_0 \subseteq_{fin} T$.
- To create a model $\mathcal{M}$ of theory T_0 it is sufficient to construct an equivalence relation on some subset of $\mathbb{N}$ which has K_2 different equivalence classes each of which contain K_1 elements.
- But obviously such a model is not a model of T. Applying **Extended Theorem** to T we obtain that T cannot be reduced to any finite theory.

Example (2)

- Let us consider a theory T of a vector space over $\mathbb{R}$.
- To encode such a vector space we need to consider each multiplication by a scalar from $\mathbb{R}$ to be a function in the model.

Example (2) continued

- $\forall u \forall v \forall w : u + (v + w) = (u + v) + w$,
- $\forall u \forall v : u + v = v + u$,
- $\forall u : 0 + u = u$,
- $\forall u \exists v : u + v = 0$,
- $\phi_a = \forall u \forall v : f_a(u + v) = f_a(u) + f_a(v)$, where $a \in \mathbb{R}$,
- $\psi_{a,b} = \forall u : f_{a+b}(v) = f_a(v) + f_b(v)$, where $a, b \in \mathbb{R}$,
- $\eta_{a,b} = \forall u : f_{a \cdot b}(u) = f_a(f_b(u))$, where $a, b \in \mathbb{R}$,
- $\forall u : f_1(v) = v$, where $1 \in \mathbb{R}$.

Example (2) continued

- Let us notice that taking a proper, countable subset $A \subsetneq \mathbb{R}$ such that $|A| = \aleph_0$.
- Then let's consider theory T_A consisting of only such $\phi_a, \psi_{a,b}$ and $\eta_{a,b}$ so $a, b, (a+b), (a \cdot b) \in A$.
- T_A has a model $\mathcal{M}_A$ which is simply a restriction of the whole $\mathbb{R}$ to its subset A performing an action on the family of vectors.
- It is important to notice that such a restriction does not have to be a field itself.
- Hence $\mathcal{M}_A$ need not to be a model of T.
- It is always possible to construct such a model assigning f_c to be an identity function, for all $c \in \mathbb{R} \setminus A$.
- Then $\mathcal{M}_A$ is not a vector space.

Example (2) continued

- Having that for an arbitrary subset A, we arrive at premises of the **Extended Theorem**.
- After applying it we obtain that theory T cannot be reduced to any theory T' such that $|T'| < |T|$.

Literature

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Thank you for your attention !