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TACL Summer School

My mini course on Coalgebra

Krakow, 21-23.7.26

Classes:

I. Tuesday, 21.7.26, 10-11:30

Categories - the absolute basics
Coalgebras
Semantics

II. Wednesday, 22.7.26, 10-11:30

Trace Semantics
via Monads & Algebras

III. Thursday, 23.7.26, 10-11:30

Logics, games, Distances
via Liftings

I. CATEGORIES (BASICS)
COALGEBRA
(STRONG) SEMANTICS

X is an object

A category $\mathcal{C}$ is a collection of objects $X \in obj(\mathcal{C})$ and a collection of arrows $f : X \to Y \in arr(\mathcal{C})$.

We write $Hom(X,Y) = \{ f \mid f : X \to Y \}$

f is an arrow from X to Y

such that

1. For each $X \in obj(\mathcal{C})$, there exists an arrow (called identity arrow on X) $id_X : X \to X$.
2. For each $f : X \to Y$, $g : Y \to Z$, there exists a composed arrow $g \circ f : X \to Z$.
3. Composition of arrows is associative, i.e.,
$$h \circ (g \circ f) = (h \circ g) \circ f$$ ..... whenever one of the sides is defined
4. id_X is neutral for composition, i.e.,
$$f \circ id_X = f = id_Y \circ f$$

Examples:

1. **Set** is the category of sets and functions.
2. A monoid is an algebraic structure $(M, +, 0)$ here in additive notation, with + - binary, 0 - constant such that + is associative and 0 is a unit.

Monoids, together with monoid homomorphisms form a category, **Mon**.

$(M_1, +_1, 0_1)$, $(M_2, +_2, 0_2)$ - monoids

$h : M_1 \to M_2$ with

$h(x+y) = h(x) + h(y)$, $h(0_1) = 0_2$

3. **Pos** is the category of posets, i.e., partially ordered sets $(P, \leq)$ with $\leq$ reflexive, antisymmetric, and transitive

and monotone maps $h: (P_1, \leq_1) \to (P_2, \leq_2)$ a function $h: P_1 \to P_2$ with $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$.

4. A particular poset $(P, \leq)$ forms a category $\mathbb{P}$ with $obj\,\mathbb{P} = P$ and $f: x \to y \in arr(\mathbb{P})$ iff $x \leq y$ in $\mathbb{P}$.

Identity arrows exist by reflexivity and composition by transitivity.

This is a very specific small category in which we have a unique arrow (if at all) from an object to another.

<u>Def.</u> Given $\mathbb{C}, \mathbb{D}$ - categories, a functor $F: \mathbb{C} \to \mathbb{D}$ assigns to every $X \in obj(\mathbb{C})$ an object $FX \in obj(\mathbb{D})$ and to every $f: X \to Y \in arr(\mathbb{C})$ an arrow $Ff: FX \to FY \in arr(\mathbb{D})$ such that identities and compositions are preserved, i.e.,

$$F id_X = id_{FX}, \text{ for all } X \in obj(\mathbb{C})$$

$$F(g \circ f) = Fg \circ Ff, \text{ for all } f: X \to Y, g: Y \to Z \in arr(\mathbb{C})$$

<u>Examples</u>

1. $Id: \mathbb{C} \to \mathbb{C}$ with $Id(X) = X$, $Id(f) = f$ is the identity functor

2. For a fixed object $A \in Obj(\mathcal{C})$, we get a constant functor with $A(X) = A$, $A(f) = id_A$ for all $X \in obj(\mathcal{C})$, $f \in arr(\mathcal{C})$ (3)

3. In a category $\mathcal{C}$ with products $A \times Id$ is a functor, also denoted $A \times -$ with $(A \times Id)(X) = A \times X$ and for $f: X \to Y$, $(A \times Id)(f) = \underbrace{id_A \times f}_{\text{sometimes denoted } A \times f} : A \times X \to A \times Y$ is given by $(A \times f)(a, x) = (a, f(x))$

We will mainly deal with functors on Set, and Set has products.

4. For a fixed set A, the constant exponent functor $Id^A = (-)^A$ on Set is given by

$X^A = \{ f \mid f: A \to X \}$ and for $h: X \to Y$

$h^A : X^A \to Y^A$ with $h^A(f) = h \circ f$

5. The powerset functor on Set, $\mathcal{P}: Set \to Set$ is defined by $\mathcal{P}X = \{ u \mid u \subseteq X \}$, and for $f: X \to Y$

$\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ with $\mathcal{P}f(u) = f(u)$ – the direct image.

6. The probability distribution functor $\mathcal{D}: Set \to Set$ is defined by $\mathcal{D}X = \{ \varphi : X \to [0,1] \mid \sum_{x \in X} \varphi(x) = 1 \}$ on a set X and on arrow $f: X \to Y$, $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y$ by

$$\mathcal{D}f(\varphi)(y) = \sum_{x : f(x) = y} \varphi(y)$$

Cat is the category of all small categories and functors between them. $\rightsquigarrow$ of course, functors compose.

One level up, there is also a way to transform functors to functors, by a so-called natural transformation. 4

<u>Def.</u> Let $\mathcal{C}, \mathbb{D}$ be categories and $F, G : \mathcal{C} \to \mathbb{D}$ functors.
A natural transformation ν from F to G, notation
$\nu : F \Rightarrow G$ (or $\nu : F \to G$) is a family of arrows
$\nu_X : FX \to GX$ in $\mathbb{D}$ indexed by $obj(\mathcal{C})$
such that for every $f : X \to Y$ in $\mathcal{C}$, the following (naturality) diagram commutes:

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\nu_X"'] & FY \arrow[d, "\nu_Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd} \qquad \text{i.e. } \nu_Y \circ Ff = Gf \circ \nu_X$$

This gives us the absolute basics of category theory.

We now turn to coalgebra, but first (as a motivation for the terminology) let's briefly discuss algebras.

One can think of an algebra for a functor $F : \mathcal{C} \to \mathcal{C}$ as a pair (A, a) of an object A – the carrier, and an arrow $a : FA \to A$ – the algebraic structure.

<u>Example:</u> Let F be the (polynomial) functor on Set defined by $FX = X \times X + 1$
Here $\times$ is product, $+$ is coproduct which on Sets corresponds to disjoint union and $1 = \{*\}$ is a singleton → the final object in Set
Then an F-algebra with carrier A is an arrow
$a : X \times X + 1 \to X$
or, equivalently, pair of functions $a_1 : X \times X \to X$
$a_2 : 1 \to X$

hence a binary operation a_1 and a constant $a_2(*)$.

An F-algebra homomorphism from $a: FA \to A$ to $b: FB \to B$ is an arrow $h: A \to B$ that makes the following diagram commute

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "a"'] & FB \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

i.e. $b \circ Fh = h \circ a$

F-algebras and their homomorphisms form a category $Alg(F)$.

No equations in this notion, but indeed any class of classical algebras without equations (given by a signature) is an algebra of a polynomial functor.

These algebras for a functor are not particularly interesting for us. What is interesting is that a coalgebra is their dual notion.

<u>Def</u>. Let $\mathcal{C}$ be a category and $F: \mathcal{C} \to \mathcal{C}$ a functor. An F-coalgebra is a pair (X, c) with $c: X \to FX$. (we often call just $c: X \to FX$ a coalgebra).

<u>Def</u>. Given two F-coalgebras $(X,c), (Y,d)$ with $c: X \to FX$, $d: Y \to FY$ a homomorphism from (X,c) to (Y,d) is an arrow $h: X \to Y$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c"'] & Y \arrow[d, "d"] \\ FX \arrow[r, "Fh"'] & FY \end{tikzcd}$$

i.e. $d \circ h = Fh \circ c$.

We call X – the state space and c – the transition structure.

Then a homomorphism is a function that "preserves and reflects the transitions."

Examples (we stick to Set here) $X = \{x, y, z\}$

1. An Id – coalgebra is just a map $c : X \to X$

one can see it as a deterministic transition system (deterministic graph) with $c(x)$ being the unique next state that x transitions to.

Here we write and draw $u \to v$ for $v = c(u)$.

ex.

$$\begin{tikzcd} x \arrow[r] & y \arrow[r] & z \arrow[loop right] \end{tikzcd}$$

2. For a constant set A, an $A \times \mathrm{Id}$ – coalgebra is a map $c : X \to A \times X$ representing a "stream" – we will clarify this later.

We write and draw $x \xrightarrow{a} y$ for $c(x) = (a, y)$ and see that such a coalgebra is a deterministic labelled transition system where in every state there is a unique action $a \in A$ which the state "performs" and "moves" to a unique successor state.

ex.

$$\begin{tikzcd} x \arrow[r, "a"] & y \arrow[r, bend left, "b"] & z \arrow[l, bend left, "a"] \end{tikzcd} \qquad A = \{a, b\}$$

3. For $F = (-)^A = \mathrm{Id}^A$, an F-coalgebra is an input enabled deterministic transition system $c : X \to X^A$.

Again we write $x \xrightarrow{a} y$ now for $c(x)(a) = y$.

So, for every action $a \in A$, $x \xrightarrow{a} c(x)(a)$ (hence, it is input-enabled).

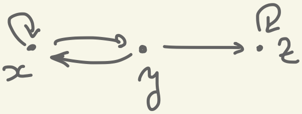
ex. $A = \{a, b\}$

4. A $\mathcal{P}$-coalgebra on Set is a non-deterministic transition system $c : X \to \mathcal{P}X$ (equivalently $c \subseteq X \times X$)

Just a directed graph

and we write $x \to y$ for $y \in c(x)$.

ex.

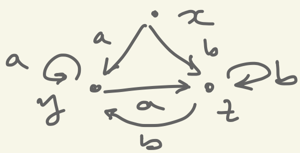


5. A labelled transition system $(X, \to \subseteq X \times A \times X)$ is a coalgebra of $\mathcal{P}^A$ or equivalently of $\mathcal{P}(A \times -)$.

Hence $c : X \to (\mathcal{P}X)^A \cong \mathcal{P}(A \times X)$. $\to$ or $(a,y) \in c(x)$

We write $x \xrightarrow{a} y$ for $y \in c(x)(a)$

ex.



6. A deterministic automaton $(Q, \Sigma, \delta : Q \times \Sigma \to Q, F)$ without specified initial state is a coalgebra of type $2 \times (-)^A$

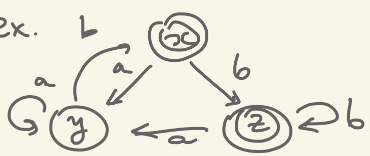
here $2 = \{0,1\}$ is a 2-element set

$$Q \xrightarrow{c} 2 \times Q^A$$

with $\pi_1(c(x)) = \begin{cases} 0, & x \notin F \\ 1, & x \in F \end{cases}$

$$\pi_2(c(x))(a) = y \quad \text{iff} \quad x \xrightarrow{a} y.$$

ex.



as usual, we draw double circles to denote accepting states.

Q: How about NA?

STRONG SEMANTICS

When do two states in an F-coalgebra, or in two F-coalgebras behave the same?

There exist several different (or seemingly different) yet canonical notions of semantics – we will look at at least three of them.

I. Bisimulation and Bisimilarity

Def. (Park, Milner) Given an LTS $(X, \to_X \subseteq X \times A \times X)$ equivalently a coalgebra $X \to (\mathcal{P}X)^A$, and another one $(Y, \to_Y \subseteq Y \times A \times Y)$, i.e., $Y \to (\mathcal{P}Y)^A$, a relation

$$R \subseteq X \times Y \quad \text{is a bisimulation iff}$$

$$\forall x \in X, y \in Y.\ (x,y) \in R \Rightarrow \forall a \in A.$$

$$① \ \forall x' \in X.\ x \xrightarrow{a} x' \Rightarrow \exists y' \in Y.\ y \xrightarrow{a} y' \wedge (x',y') \in R$$

$$\wedge$$

$$② \ \forall y' \in Y.\ y \xrightarrow{a} y' \Rightarrow \exists x' \in X.\ x \xrightarrow{a} x' \wedge (x',y') \in R$$

This is a very well studied notion – discuss..

We say that x is bisimilar to y and write $x \sim y$ iff there exists a bisimulation R with $(x,y) \in R$.

Here we move to the slides ... which also give us "a way toward" coalgebraic bisimilarity.

Def [Aczel & Mendler] A coalgebraic bisimulation from an F-coalgebra $(X, c : X \to FX)$ to an F-coalgebra $(Y, d : Y \to FY)$

is a relation $R \subseteq X \times Y$ such that there exists a transition structure $r : R \to FR$ making both canonical projections $\pi_1 : R \to X$, $\pi_2 : R \to Y$ coalgebra homomorphisms.

In a diagram

$$\begin{tikzcd} X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "r"] & Y \arrow[d, "d"] \\ FX & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FY \end{tikzcd}$$

$x \sim_{AM} y$ iff there exists a coalgebraic bisimulation relating them.

Equivalently, one can define a bisimulation to be a relation as a jointly monic span $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ with $\langle r_1, r_2 \rangle$ mono (injective in Sets).

II. Final coalgebra semantics

Sorry - we need one more notion from category theory.

Def. A final object in a category $\mathcal{C}$ is an object denoted by 1 such that for all $X \in obj(\mathcal{C})$ there exists a unique arrow $!_X : X \to 1$. Final objects not always exist.

Set has a terminal object, indeed 1 (any singleton) unique up-to isomorphism.

If a final coalgebra (Z,ξ) exists in $\mathrm{CoAlg}(F)$, then it provides canonical semantics too.

Let (X,c), (Y,d) be two F-coalgebras, and assume that (Z,ξ) is the final F-coalgebra.
Let $beh_c : X \to Z$ and $beh_d : Y \to Z$ be the unique coalgebra homomorphisms that exist by finality.
We say that $x \in X$ and $y \in Y$ are equivalent in the final coalgebra semantics, and write $x \sim_{f.c.} y$ iff

$$beh_c(x) = beh_d(y).$$

Lemma (Lambek) If (Z,ξ) exists in $\mathrm{CoAlg}(F)$, then ξ is iso.

Proof – nice excercise

Assume (Z,ξ) exists. Then we have the following situation

$$\begin{tikzcd}
Z \arrow[r, "\xi"] \arrow[d, "\xi"'] \arrow[rr, dashed, bend left, "beh_\xi"] \arrow[dr, phantom, "\textcircled{1} ="] & FZ \arrow[r, dashed, "beh_{F\xi}"] \arrow[d, "F\xi"'] \arrow[dr, phantom, "\text{finality } \textcircled{2}"] & Z \arrow[d, "\xi"] \\
FZ \arrow[r, "F\xi"'] \arrow[rr, dashed, bend right, "F beh_\xi"'] & FFZ \arrow[r, dashed, "F beh_{F\xi}"'] & FZ
\end{tikzcd}$$

Now:

* $beh_\xi = id_Z$ as id_Z is a coalg. hom. from (Z,ξ) to (Z,ξ)

* from ① and ② we have that

$beh_{F\xi} \circ \xi$ is a coalgebra homomorphism

from (Z, ξ) to (Z, ξ). So this must be the unique one i.e.

$$beh_{F\xi} \circ \xi = id_Z$$

* finally $\xi \circ beh_{F\xi} = F beh_{F\xi} \circ F\xi = F(beh_{F\xi} \circ \xi) =$

$$= F(id_Z) = id_{FZ}.$$

As a consequence, it is often clear that a final coalgebra doesn't exist by simple cardinality arguments.

For example, by the small Theorem of Cantor, there is no final $\mathcal{P}$-coalgebra (as there is no iso $X \to \mathcal{P}X$).

What we also see about final coalgebra semantics is that we are in a situation where we have denotational semantics. Z contains "all possible behaviors" and for $(X, c: X \to FX)$ and $x \in X$, $beh_c(x)$ is the denotation / the behavior of state x. of course in Set or other concrete cat...

Example: To illustrate this, let's go back to the "streams" that I promised to mention / explain. ...... $\to$

We consider $F = A \times (-)$ on Set.

This functor has a final coalgebra, given by the set of all infinite words / streams on A

$$Z = A^{\omega} \quad \text{and} \quad \zeta : Z \xrightarrow{\cong} A \times Z$$

is $\zeta : A^{\omega} \longrightarrow A \times A^{\omega}$ given by $\langle head, tail \rangle$

$$\zeta(\sigma) = (head(\sigma), tail(\sigma)).$$

if you want more precisely $\sigma : \mathbb{N} \to A$

$head(\sigma) = \sigma(0)$

$tail(\sigma) = \sigma' : \mathbb{N} \to A$

with $\sigma'(n) = \sigma(n+1), \forall n \in \mathbb{N}$

ex. Back to our little example

(X, c)
$$\begin{tikzcd} x \arrow[r, "a"] & y \arrow[r, bend left, "b"] & z \arrow[l, "a"] \end{tikzcd}$$

$beh_c(x) = a(ba)^{\omega} = ab(ab)^{\omega}$

$beh_c(y) = (ba)^{\omega}$

$x \sim_{f,c} z \quad \Longleftarrow \quad beh_c(z) = beh_c(x)$

III. Behavioural equivalence

Given two F-coalgebras (X, c) and (Y, d), a cocongruence is a cospan $(\mathcal{U}, h_X, h_Y)$ with $h_X : X \to \mathcal{U}$, $h_Y : Y \to \mathcal{U}$ such that there exists a coalgebra structure $u : \mathcal{U} \to F\mathcal{U}$ making h_X and h_Y coalgebra homomorphisms.

sometimes jointly epi-assumed

The pullback of a cocongruence is called a kernel bisimulation, since [Staton 2009 CALCO].

The categorically minded will know what a pullback is – the limit of the cospan. In set it is a relation

$$R = \{(x,y) \mid h_X(x) = h_Y(y)\}.$$

Two states $x \in X$, $y \in Y$ are called behaviourally equivalent, notation $x \sim_{b.e.} y$ if they are related

this is not used ... discuss

by some kernel bisimulation, i.e., iff there exists a cocongruence (U, h_X, h_Y) with $h_X(x) = h_Y(y)$.

This is a useful notion as it allows to speak of "equal behaviour" even if a "domain of all behaviours", that is, a final coalgebra doesn't exist.
The notion was first introduced in the context of coalgebraic modal logics.

$\boxed{\text{Theorem:}}$ For well behaved (= weak pullback preserving) functors on Set, for which final coalgebra exists

$$x \sim y \text{ iff } x \sim_{f.c.} y \text{ iff } x \sim_{b.e.} y.$$

We won't be able to prove it here, but the last equivalence should be clear:

$x \sim_{f.c.} y$, then (Z, beh_X, beh_Y) is the cocongruence (not epi ☺)

see [Bartels, S., de Vink '03] or [Staton '09] if $x \sim_{b.e} y$ and final coalgebra exists, let (14)

$(\mathcal{U}, h_X, h_Y)$ be a cocongruence witnessing this.
with transition structure u.

Then $beh_u \circ h_X$, $beh_u \circ h_Y$ are homomorphisms from (X, c), (Y, d) to (Z, ξ), respectively

and by finality $beh_c(x) = beh_u \circ h_X(x)$

$$= beh_u(h_X(x))$$
$$= beh_u(h_Y(y))$$
$$= beh_u \circ h_Y(y)$$
$$= beh_d(y)$$

Hence $x \sim_{f.c.} y$. $\square$

Now here is some reading material:

[Rutten 2000] Jan Rutten, Universal Coalgebra: A theory of systems, TCS, 2000

[Jacobs-book] Bart Jacobs, Introduction to Coalgebra (book)

[Bartels, S., de Vink '03] Falk Bartels, Ana S., Erik de Vink, A hierarchy of prob. system types.

[Staton09] Sam Staton, Relating different notions of coalgebraic bisimulations 2009 CALCO

already mentioned

[S. 05] Ana S., Probabilistic systems coalgebraically PhD thesis, Chapter 3 – intro

some students like it

Before we proceed – one more coalgebraic notion:

15

$\Sigma \rightarrow$ also well-rooted in classical results for LTS or probabilistic bisimulation.

↓ speaking of that, back to the slides again

A relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$

(i.e., jointly monic span) is a

Hermida-Jacobs bisimulation iff there

exists $r: R \to \bar{F}R$ – also denoted $Rel(F)(R)$

such that

$$\begin{tikzcd} X \arrow[d, "c"'] & R \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[d, dashed, "r"] & Y \arrow[d, "d"] \\ FX & \bar{F}R \arrow[l] \arrow[r] & FY \end{tikzcd}$$

is the relation lifting of R

our first lifting!

$\bar{F}R$ is the image of the composite

$$\bar{F}R \xrightarrow{F\langle r_1, r_2\rangle} F(X \times Y) \xrightarrow{Fr_1 \times Fr_2} FX \times FY$$

if we are thinking of standard relations (with projections) on Set

$$\bar{F}R \subseteq FX \times FY$$

and now this is another nice formulation (also explaining the AM-corr. in well behaved cases):

$$(\alpha, \beta) \in \bar{F}R \iff \exists \gamma \in FR.\ F\pi_1(\gamma) = \alpha \wedge F\pi_2(\gamma) = \beta$$

this witness is often called coupling of α and β, especially in probabilistic systems

[Hermida Jacobs 98]
Structural Induction and Coinduction in a fibrational setting. I&C

see [Jacobs-book]

II. TRACE SEMANTICS

22.7.26

1

We will go through most of this material on slides. I only make small notes here for some notions that are missing on the slides (and will at the end add all references)

<u>Def</u>. A monad on a category $\mathbb{C}$ is a triple (M, μ, η) where $M : \mathbb{C} \to \mathbb{C}$ is a functor and $\mu : MM \Rightarrow M$, $\eta : \mathrm{Id} \Rightarrow M$ are natural transformations that satisfy the following unit and associativity of multiplication laws.

(μ is called the multiplication, η the unit of the monad)

most commonly we just say "the monad M".

latest here we'll recall nat. transf.

$$
\begin{tikzcd}
MX \arrow[r, "M\eta_X"] \arrow[dr, equal] & MMX \arrow[d, "\mu_X"] & MX \arrow[l, "\eta_{MX}"'] \arrow[dl, equal] \\
& MX &
\end{tikzcd}
$$

$$
\begin{tikzcd}
MMMX \arrow[r, "M\mu_X"] \arrow[d, "\mu_{MX}"'] & MMX \arrow[d, "\mu_X"] \\
MMX \arrow[r, "\mu_X"'] & MX
\end{tikzcd}
$$

Finitary monads on Sets correspond to algebraic theories, i.e., varieties of algebras.

Here are some examples of our most favourite monads explicitly.

1 The powerset monad has unit $\eta_X : X \to \mathcal{P}X$ given by singleton $\eta_X(x) = \{x\}$ and multiplication $\mu_X : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$ by union

$$\mu_X(\mathcal{U}) = \bigcup \mathcal{U} = \{x \mid \exists S \in \mathcal{U}.\ x \in S\}$$

for $\mathcal{U} \subseteq \mathcal{P}X$.

2 The distribution monad $\mathcal{D}$ has unit $\eta_X : X \to \mathcal{D}X$ given by Dirac distribution $\eta_X(x) = \delta_x$ with $\delta_x : X \to [0,1]$, $\delta_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$

and multiplication ....

before we do that, let's go to "formal sum" notation for distributions.

For $\varphi \in \mathcal{D}X$, we write $\varphi = \sum_{i \in I} p_i x_i$ if

$$\varphi(x_i) = p_i, \quad \{x_i \mid i \in I\} \supseteq \operatorname{supp} \varphi$$

Then for $f : X \to Y$, $\mathcal{D}f$ has a nicer representation

$$\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y, \quad \mathcal{D}f(\varphi) = \sum_{i \in I} p_i f(x_i).$$

Using this, for $\Phi \in \mathcal{D}\mathcal{D}X$, $\Phi = \sum p_i \varphi_i$

$$\mu_X(\Phi)(x) = \sum p_i \varphi_i(x).$$

A monad M is presented by an algebraic theory (Σ, E) iff $MX \cong \mathrm{Terms}_{\Sigma,E}(X)$ (3)

Σ-terms with variables in X modulo equations in E

$\eta_X : X \to MX$ ⇝ every variable is a term

$\mu_X : MMX \to MX$ ⇝ substitution / flattening... (a term over terms is a term over variables)

For every set X, $\mu_X : MMX \to MX$ is an EM-algebra for M. It is <u>the free</u> M-algebra generated by X.

That is, for any $a : MA \to A$ and $f : X \to A$, there is a unique extension $f^\# : MX \to A$ which is an algebra homomorphism from (MX, μ_X) to (A, a)!

This is simply the Kleisli-extension (the basis of "determinisation")

$$f^\# = MX \xrightarrow{Mf} MA \xrightarrow{a} A$$

Hence monads have two sides:

1) computational
2) algebraic

Examples

	computational	algebraic
$\mathcal{P}$	nondeterminism	join semilattice with $\perp$
$\mathcal{D}$	probability	convex algebra (Barycentric)

While it is possible to do most things like determinisation / traces / ... without a presentation, having the algebraic presentation (the syntax) makes many things more convenient.

I may add stuff here after the class, in case we get to some interesting improvised/unplanned discussions there.
I will also add all references there, at the end.

III. $\boxed{\text{Logics}}$ via Liftings

→ yesterday we saw relation lifting

$$R \subseteq X \times X \rightsquigarrow \overline{F}R \subseteq FX \times FY$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad Rel(F)(R)$$

This lifts F (on sets) to a functor $\overline{F}$ on Rel.

Now we want to see how predicate lifting (and relation lifting) yield (modal) logics.
[Cirstea, Pattinson, Schröder, Kupke, ... originally Moss ...]

Main idea: φ – formula (property) of states
$\Box\varphi$ – property of successors "lifts to transitions"

However: To get this we need to find modalities and liftings that fit.

Similarity type Λ is a set of modal operators with arities

[Kupke & Pattinson – Overview 2011]

A Λ-structure is a functor F (they call it T for transitions ☺) and n-ary predicate lifting ②

$$[\![\heartsuit]\!] : (2^-)^n \Longrightarrow \underline{2^-} \circ F$$

contravariant powerset

for every n-ary $\heartsuit$

natural transformation (recall here)

$$[\![\heartsuit]\!]_X (\underbrace{U_1, \dots, U_n}_{\text{subsets of } X}) = T\text{-subset of } FX$$

naturality means that for $f : X \to Y$

$$\begin{tikzcd}
\overbrace{2^X \times \dots \times 2^X}^{n} \arrow[d, "{[\![\heartsuit]\!]_X}"'] & \overbrace{2^Y \times \dots \times 2^Y}^{n} \arrow[l, "(2^f)^n"'] \arrow[d, "{[\![\heartsuit]\!]_Y}"] \\
2^{FX} & 2^{FY} \arrow[l, "2^{Ff}"']
\end{tikzcd}$$

Λ induces a logic, i.e., a set of formulas $\mathcal{F}(\Lambda)$

$$\mathcal{F}(\Lambda) \ni \varphi, \psi ::= p \mid \varphi \wedge \psi \mid \neg \varphi \mid \heartsuit(\varphi_1, \dots, \varphi_n)$$

$p \in P$ - set of propositional variables

$\heartsuit \in \Lambda$ - n-ary

An F-model is a triple $(X, c, \pi) = M$

(X, c) - F-coalgebra, $\pi : P \to \wp X$ is a valuation

The semantics is defined inductively by $[\![-]\!] : \mathcal{F}(\Lambda) \to \mathcal{P}X$ ③

$$[\![p]\!]_M = \pi(p)$$

$$[\![\varphi \wedge \psi]\!]_M = [\![\varphi]\!]_M \cap [\![\psi]\!]_M$$

$$[\![\neg\varphi]\!]_M = X \setminus [\![\varphi]\!]_M$$

$$[\![\heartsuit(\varphi_1, \ldots, \varphi_n)]\!]_M = c^{-1} \circ [\![\heartsuit]\!]_X \left([\![\varphi_1]\!]_M, \ldots, [\![\varphi_n]\!]_M\right)$$

Hence $x \models \heartsuit(\varphi_1, \ldots, \varphi_n)$ if

$$c(x) \in [\![\heartsuit]\!]_X ([\![\varphi_1]\!]_M, \ldots, [\![\varphi_n]\!]_M)$$

Examples

① $F = \mathcal{P}$, $\Lambda = \{\Box\}$

$$[\![\Box]\!]_X(Y) = \{U \in \mathcal{P}X \mid U \subseteq Y\}$$

yields the logic K

② $F = \mathcal{P}^A$, $\Lambda = \{\Box_a \mid a \in A\}$

$$[\![\Box_a]\!]_X(Y) = \{f : A \to \mathcal{P}X \mid f(a) \subseteq Y\}$$

yields Hennessy Milner logic

③ $F = \mathcal{D}$, $\Lambda = \{L_p \mid p \in [0,1) \cap \mathbb{Q}\}$

$L_p\varphi$ has intended meaning

"φ holds with probability at least p in the successors"

$$[\![L_p]\!]_X(Y) = \{\nu \in \mathcal{D}X \mid \nu(Y) \geq p\}$$

as expected $x \models L_p \varphi$ iff
$\xi(x)([\![\varphi]\!]) \geq p$

A logic defines an equivalence of states:
(we denote it here by $\equiv_L$)

$$x \equiv_L y \quad \text{iff} \quad \forall \varphi \in \mathcal{F}(\Lambda).\ x \models \varphi \iff y \models \varphi$$

A logic is <u>adequate</u> if

$$\sim_{b.e} \;\subseteq\; \equiv_L$$

(behaviorally equivalent states are logically indistinguishable)

The shape and naturality of the predicate liftings $[\![\heartsuit]\!]$ are "exactly what is needed for adequacy".

A logic is <u>expressive</u> if

$$\equiv_L \;\subseteq\; \sim_{b.e}$$

(logically indistinguishable states are behaviourally equivalent)

<u>Def.</u> A Λ-structure is separating if for all X

$$FX \longrightarrow \prod_{\substack{\heartsuit \in \Lambda \\ n\text{-ary}}} \mathcal{P}((\mathcal{P}(X))^n)$$

$$t \mapsto \{ (u_1, .., u_n) \in (\mathcal{P}X)^n \mid t \in [\![\heartsuit]\!] (u_1, .., u_n) \}$$ (5)

is jointly injective

Elements of FX can be distinguished by lifted predicates..

Theorem: If F is finitary and Λ-separating then the logic is expressive, i.e.
behavioural equivalence and logical equivalence coincide.

Baldan Bonchi König 2018
+ ... Komorida, Katsumata, Hasuo .. (codensity lifting)

Distances

Coalgebras $X \to FX$

lifting of F to $\overline{F}$ on PMet

From $X \to FX$, $(X, disc) \to \overline{F}(X, disc)$

If a final F-coalgebra exists, one obtains from it
(Z, ς)

$$(Z, d_Z) \xrightarrow{\hat{\varsigma}} \overline{F}(Z, d_Z) \qquad \text{final } \overline{F}\text{-coalg.}$$

$$\text{beh-dist}(x,y) = d_Z(!(x), !(y))$$
$$= d_Z(\text{beh}(x), \text{beh}(y))$$

They also provide trace metrics with similar approach! after determinization