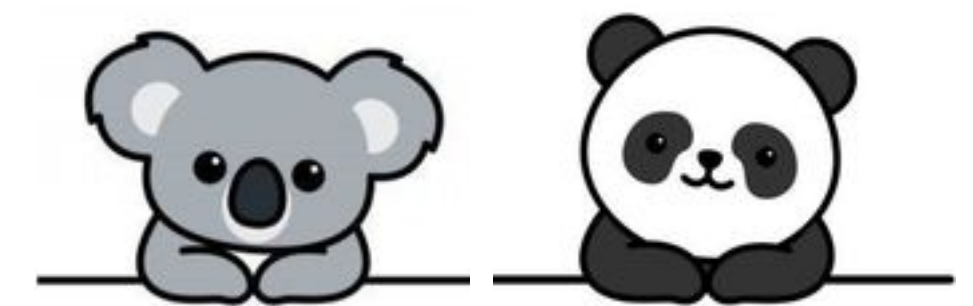


Coalgebra Mini Course



TACL 2026 School

Ana Sokolova, July 21-23, 2026, TACL @ Krakow

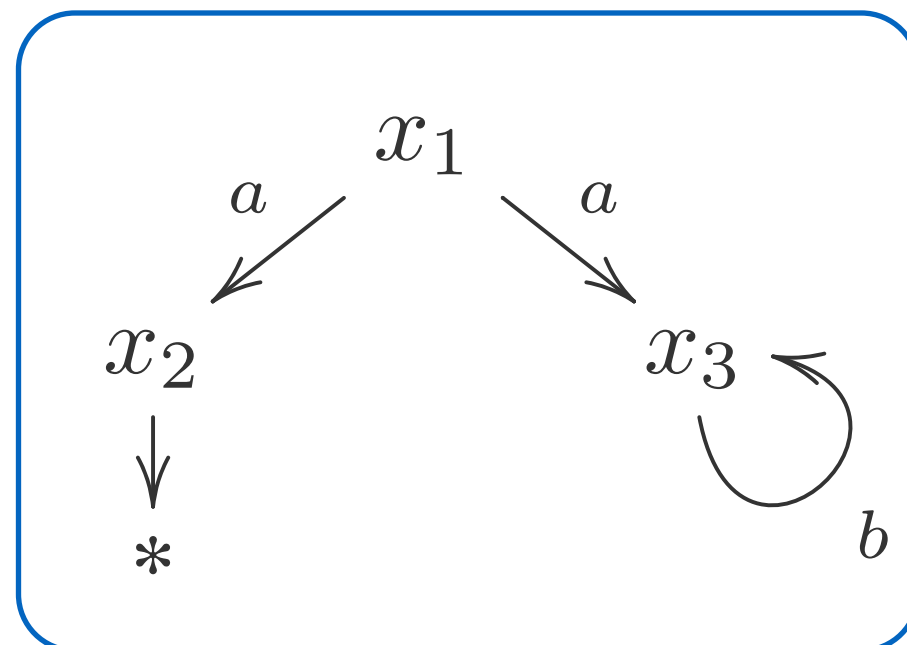
I. Coalgebras and Strong Semantics



Some coalgebras

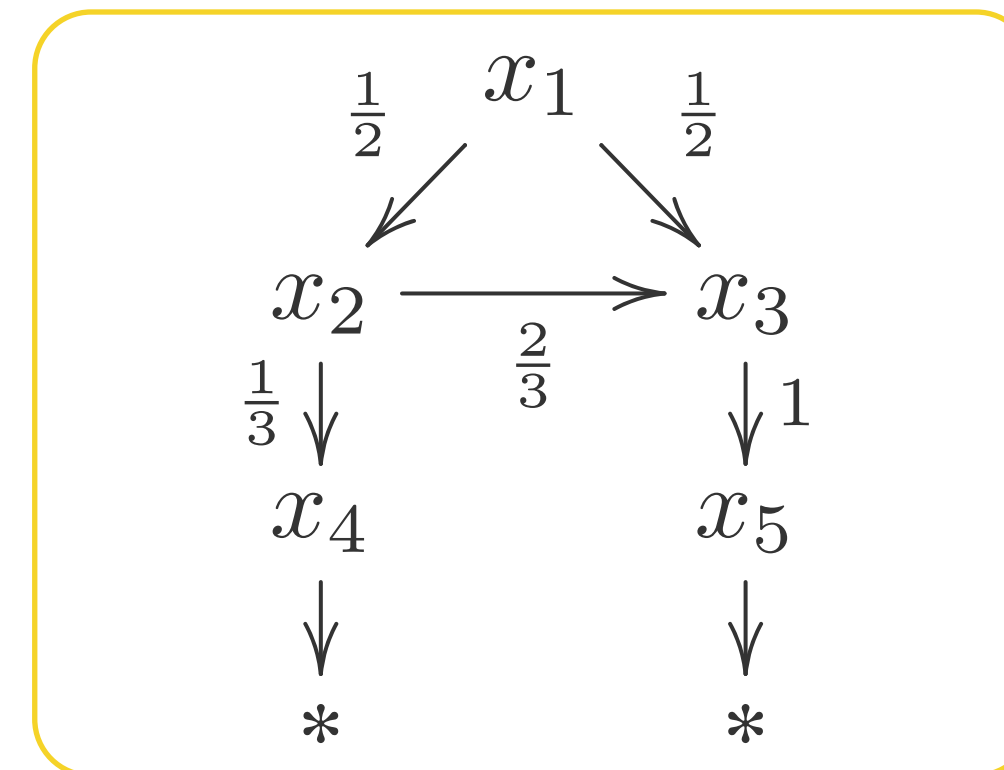
NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



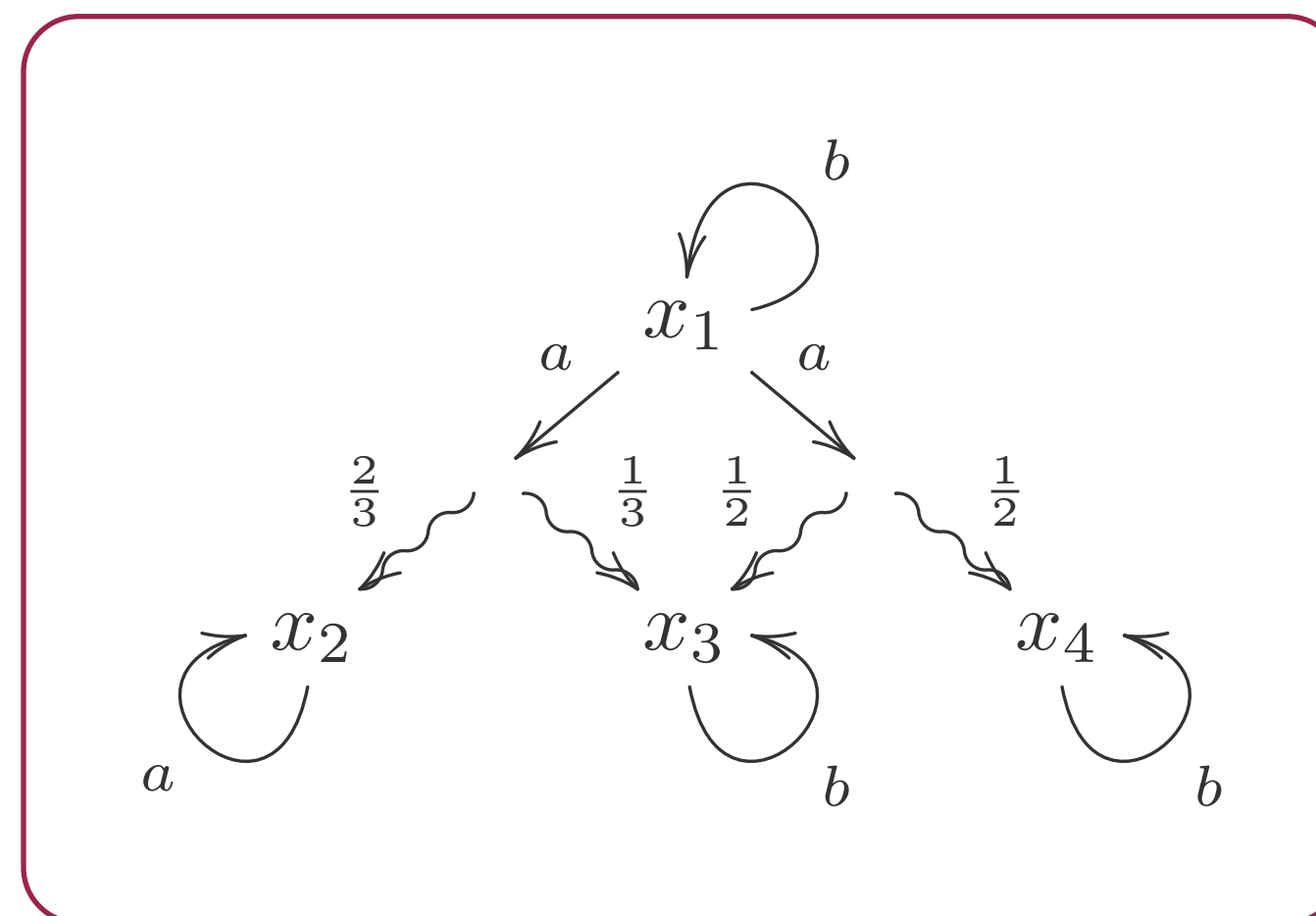
MC

$$X \rightarrow \mathcal{D}X + I$$



PA

$$X \rightarrow (\mathcal{P}\mathcal{D}X)^A$$



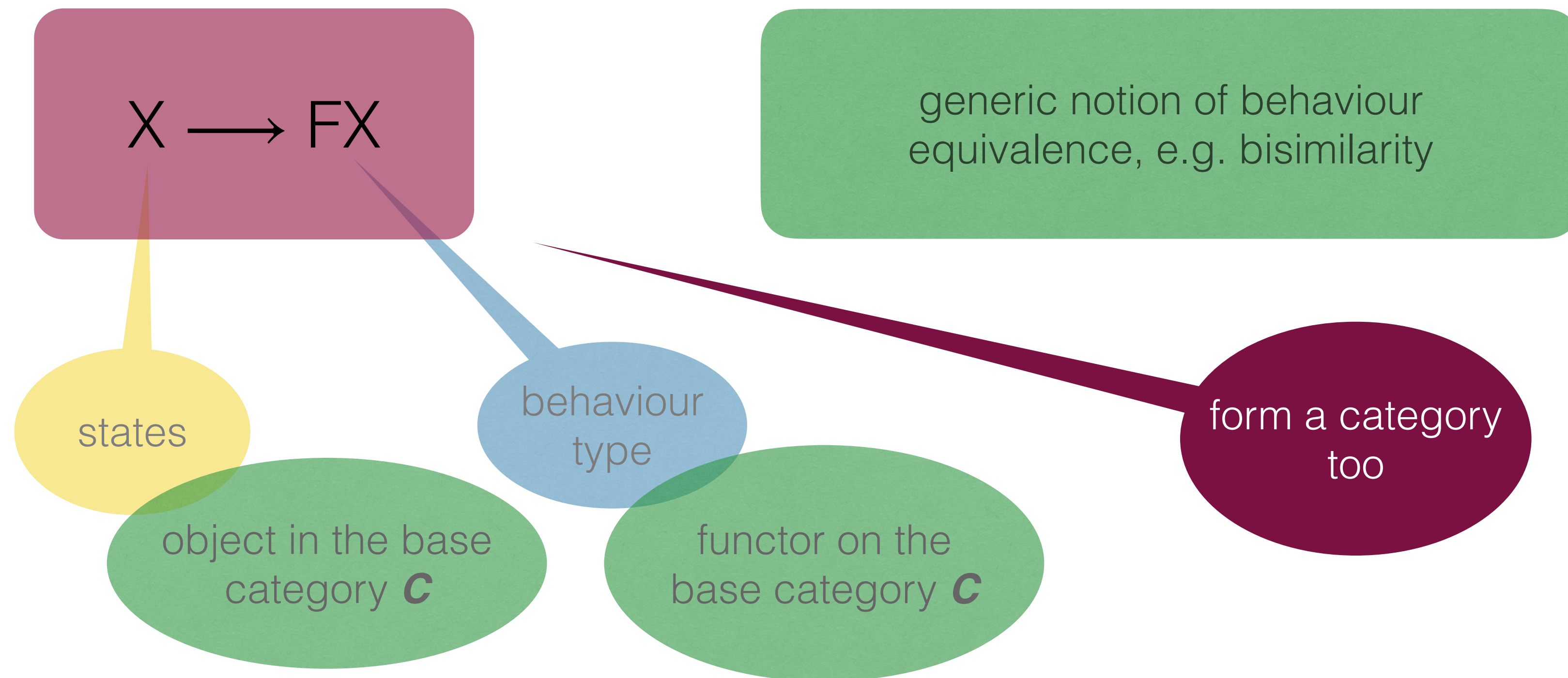
Various transitions
systems / automata are
coalgebras



Coalgebra

Uniform framework for dynamic transition systems, based on category theory.

A coalgebra is a generic transition system:



Bisimilarity of Labelled Transition Systems

$$S \rightarrow (\mathcal{P} S)^A$$

Bisimilarity of Labelled Transition Systems

An LTS is a map

$$S \rightarrow (\mathcal{P} S)^A$$

a coalgebra

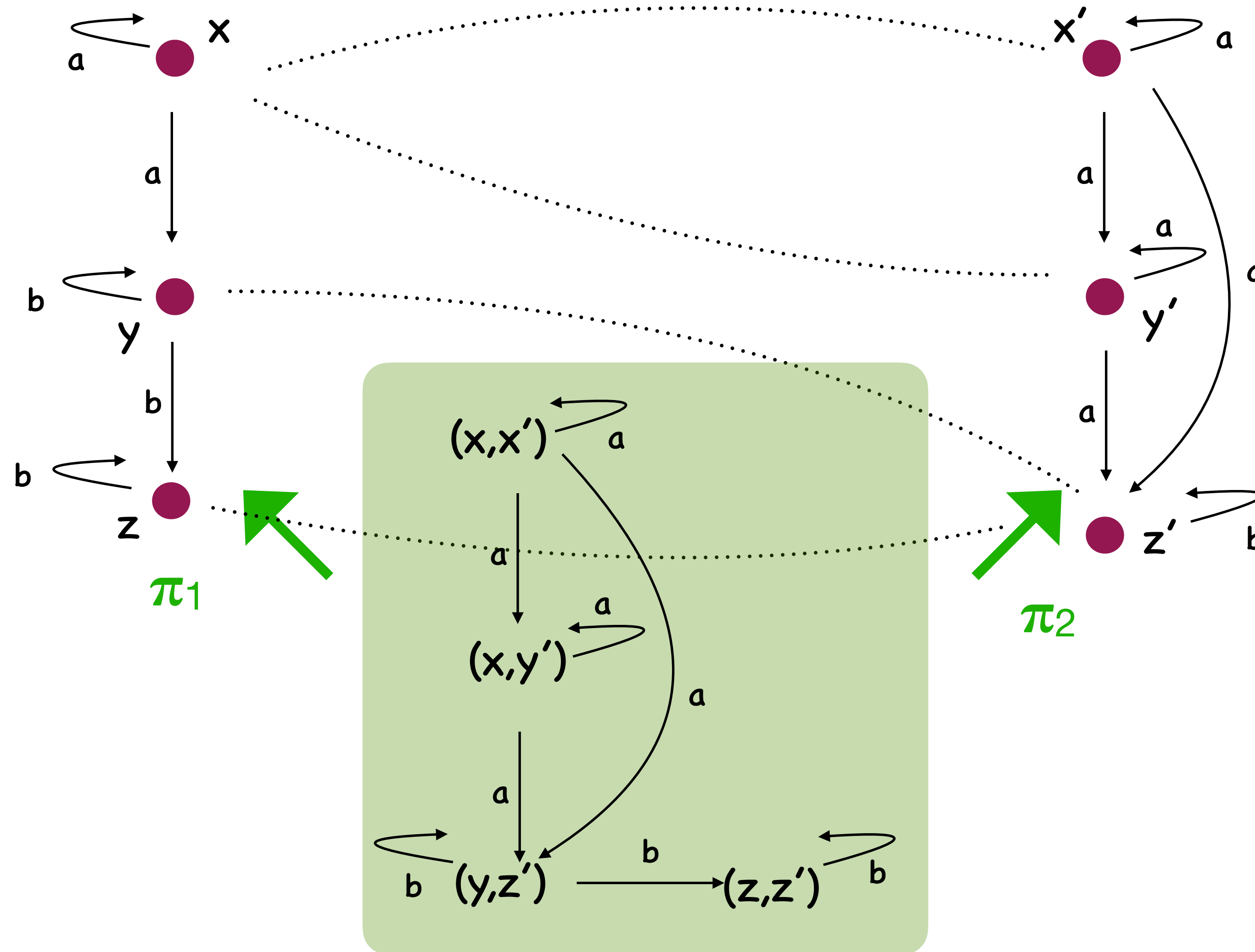
with a set of states S and a set of actions A .

A binary relation R between the states of two LTS is a bisimulation
iff whenever two states are related, each transition from one of them
has a matching transition from the other and the target states are related again.

Two states are bisimilar iff they are related by a bisimulation.

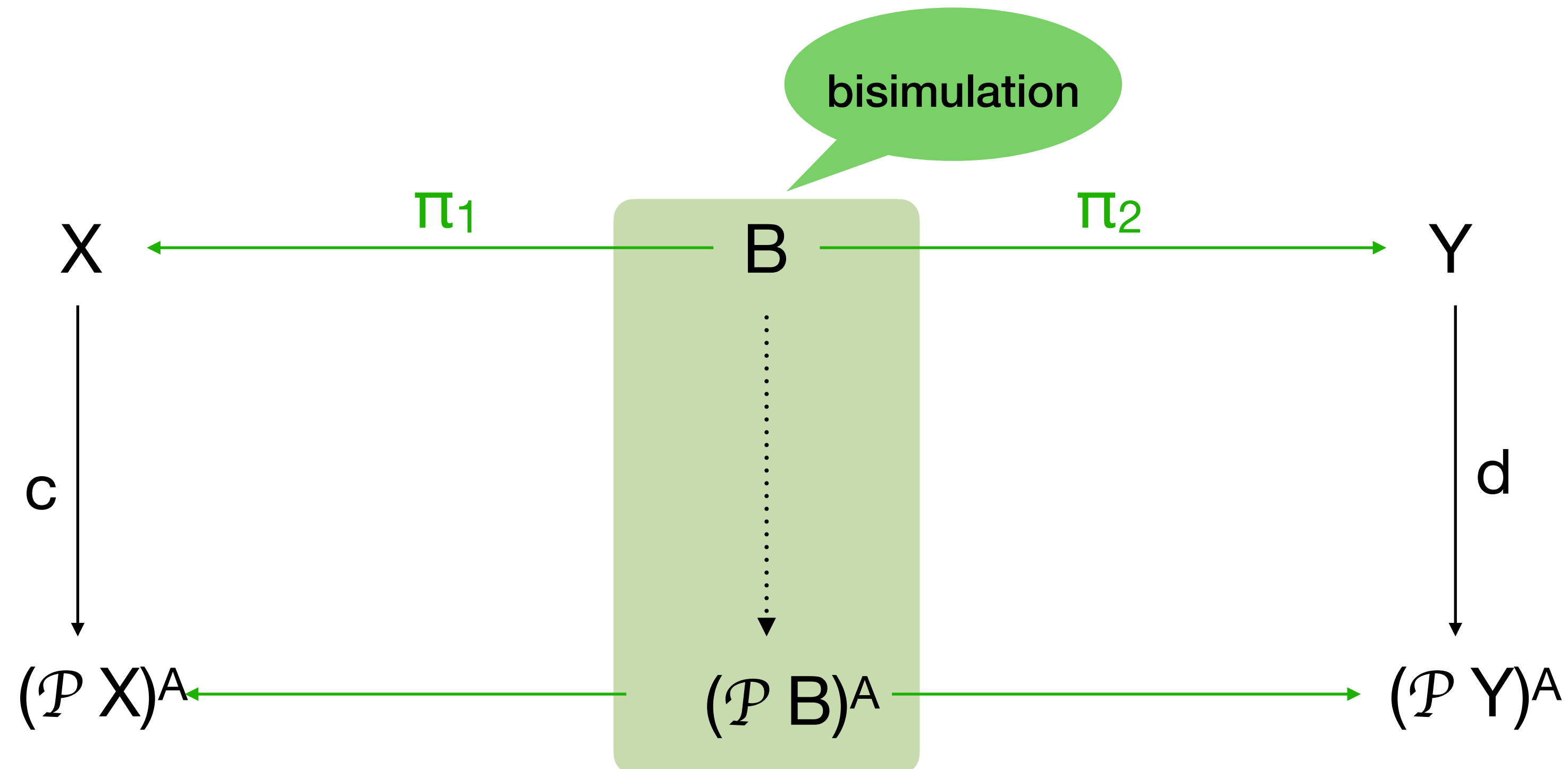
Bisimilarity of Labelled Transition Systems

$$S \rightarrow (\mathcal{P} S)^A$$

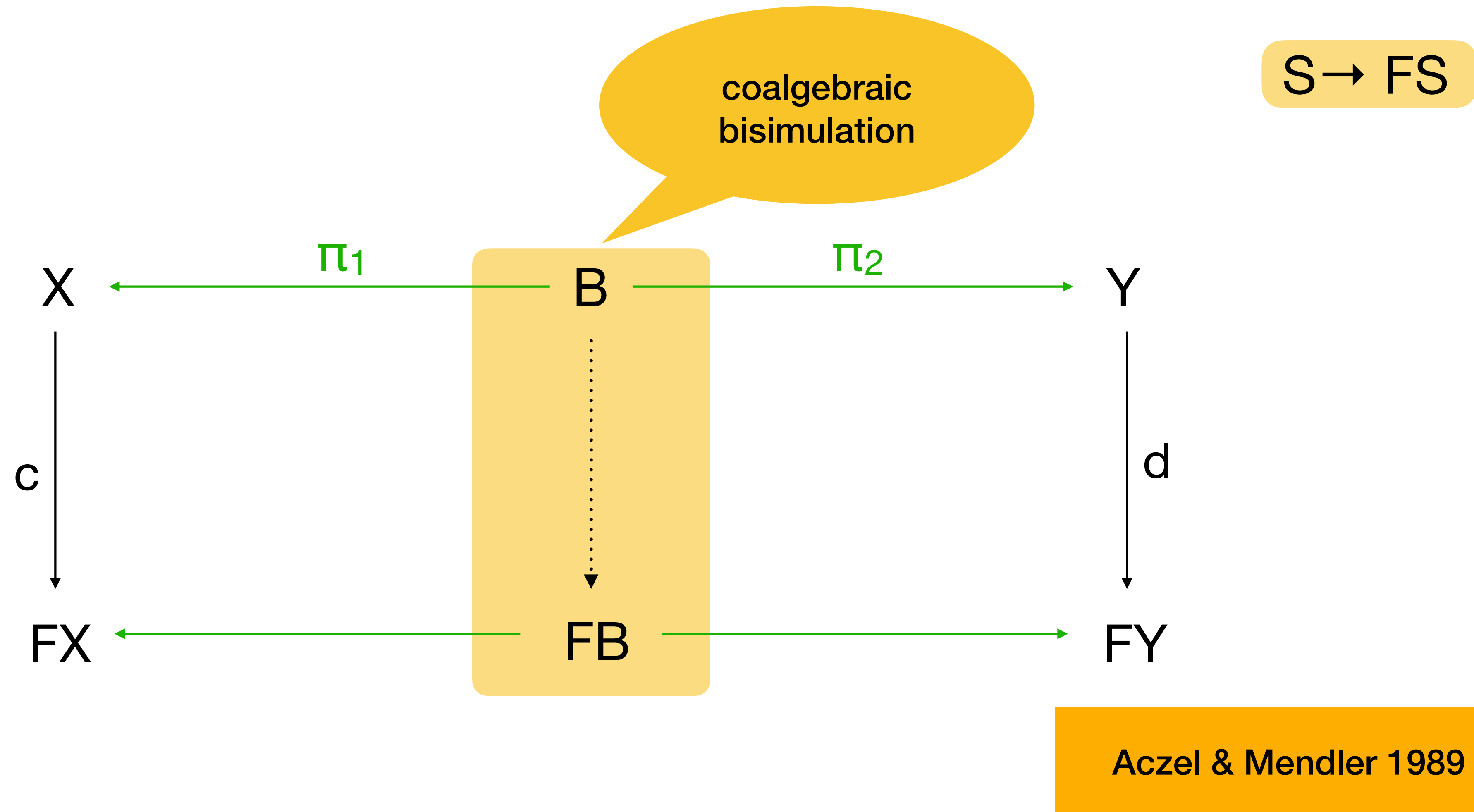


Bisimilarity of Labelled Transition Systems

$$S \rightarrow (\mathcal{P} S)^A$$



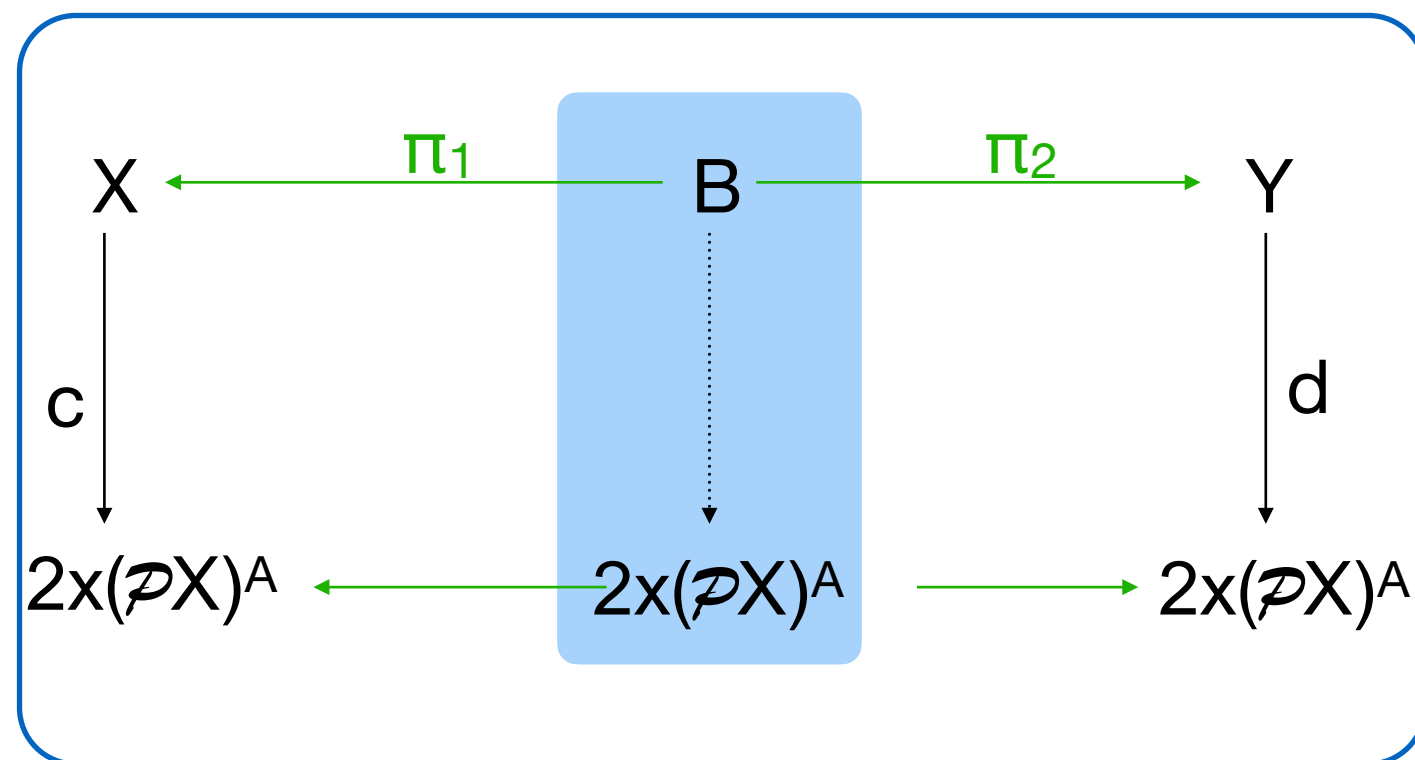
Bisimilarity of F-Coalgebras



Coalgebraic bisimilarity is bisimilarity

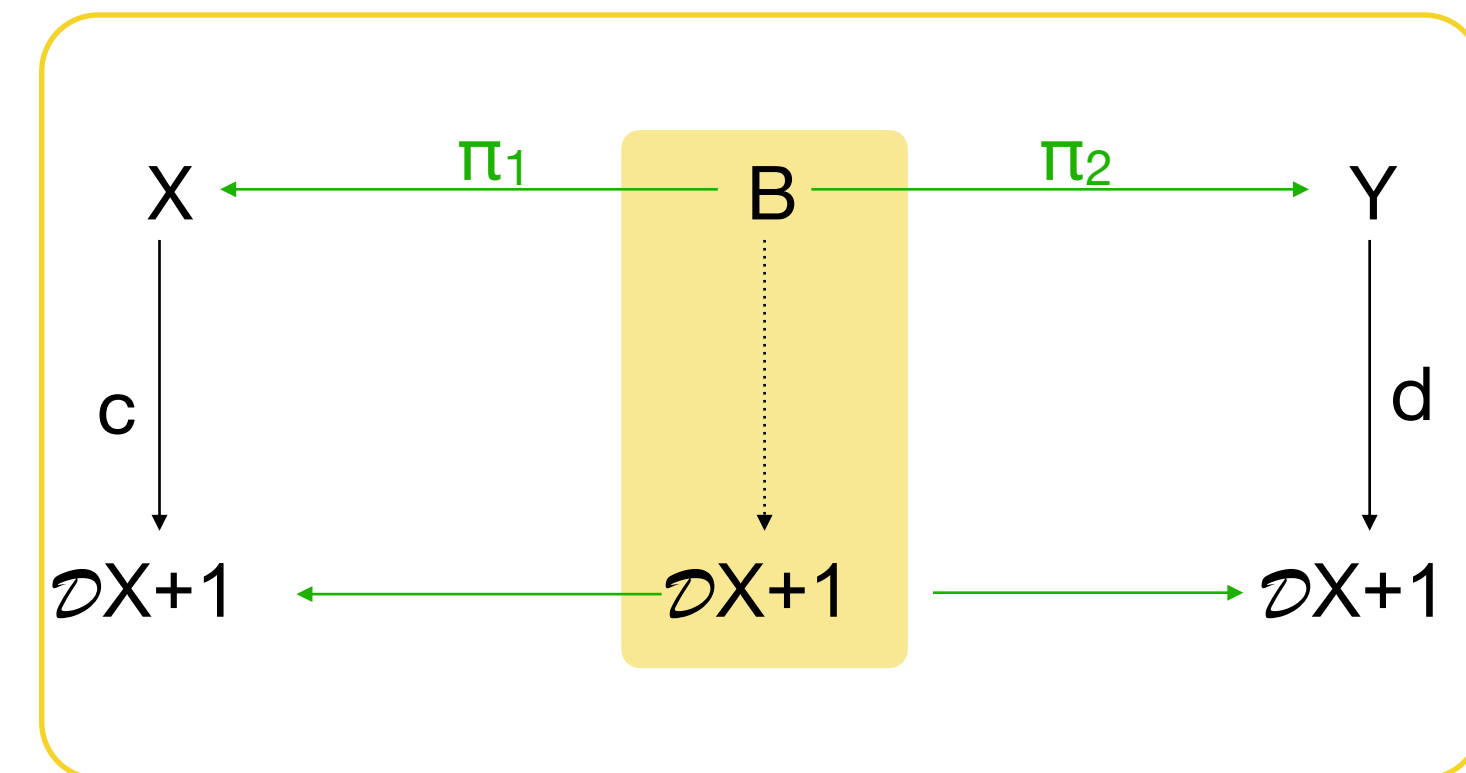
NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



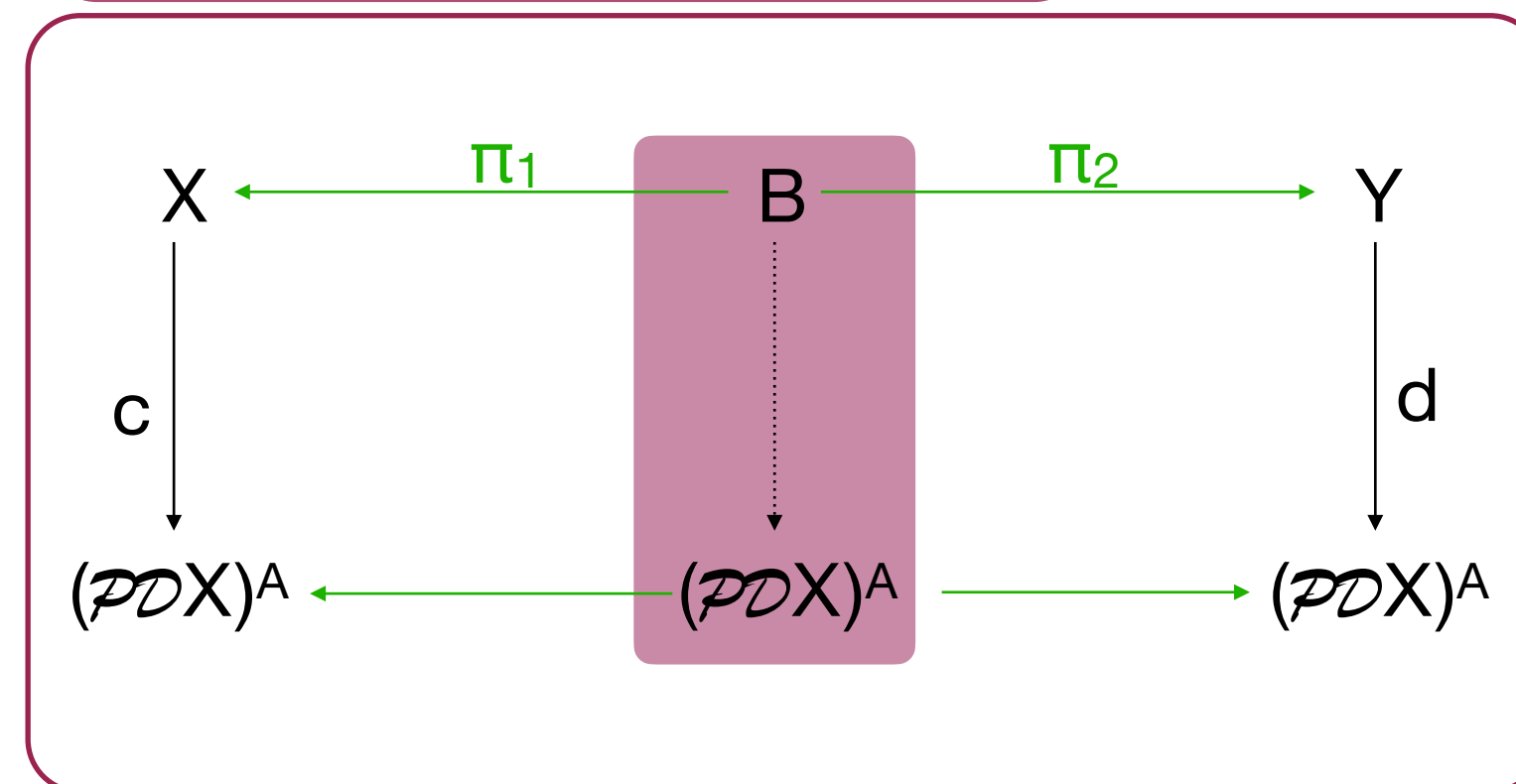
MC

$$X \rightarrow \mathcal{D}X + I$$



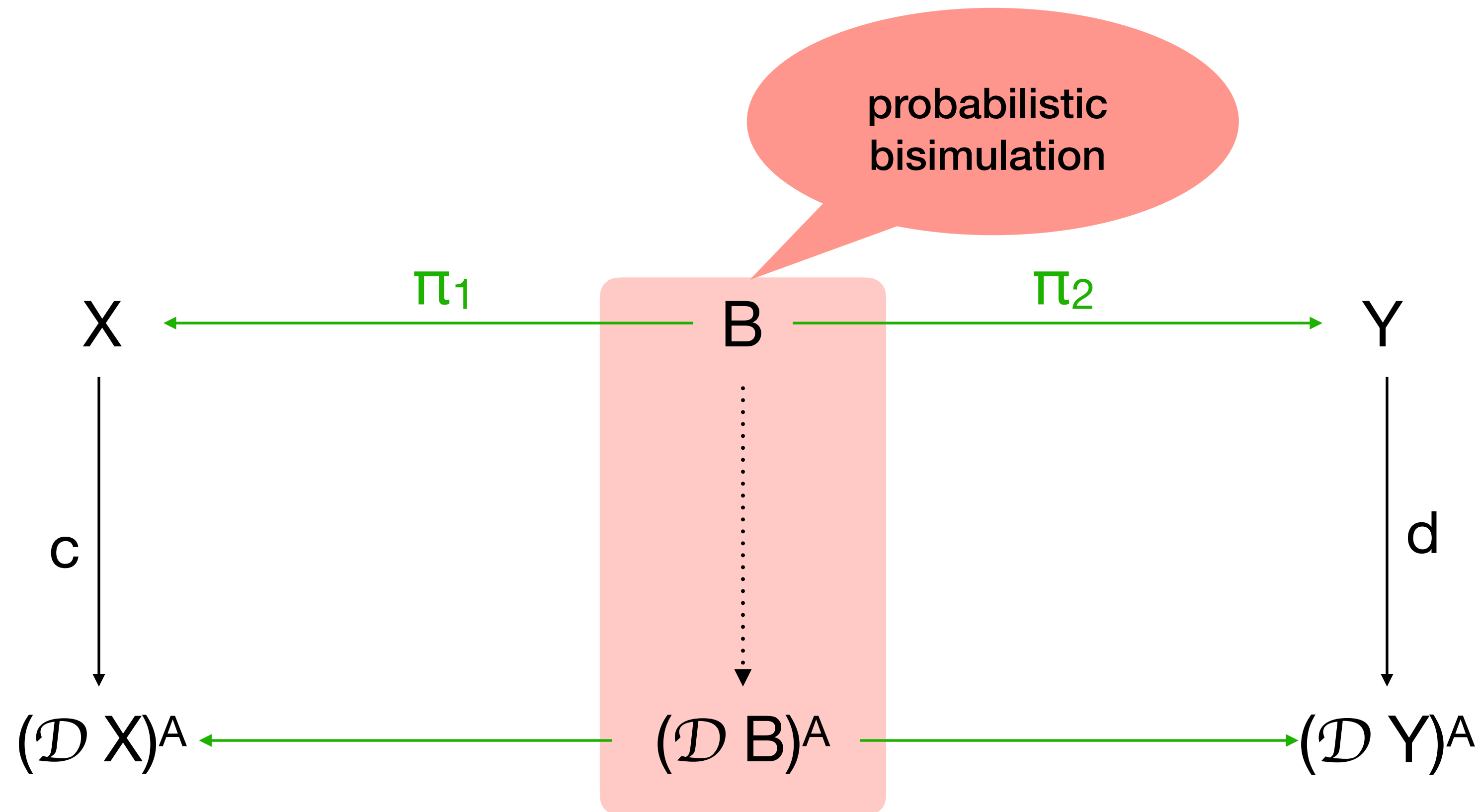
PA

$$X \rightarrow (\mathcal{P}\mathcal{D}X)^A$$

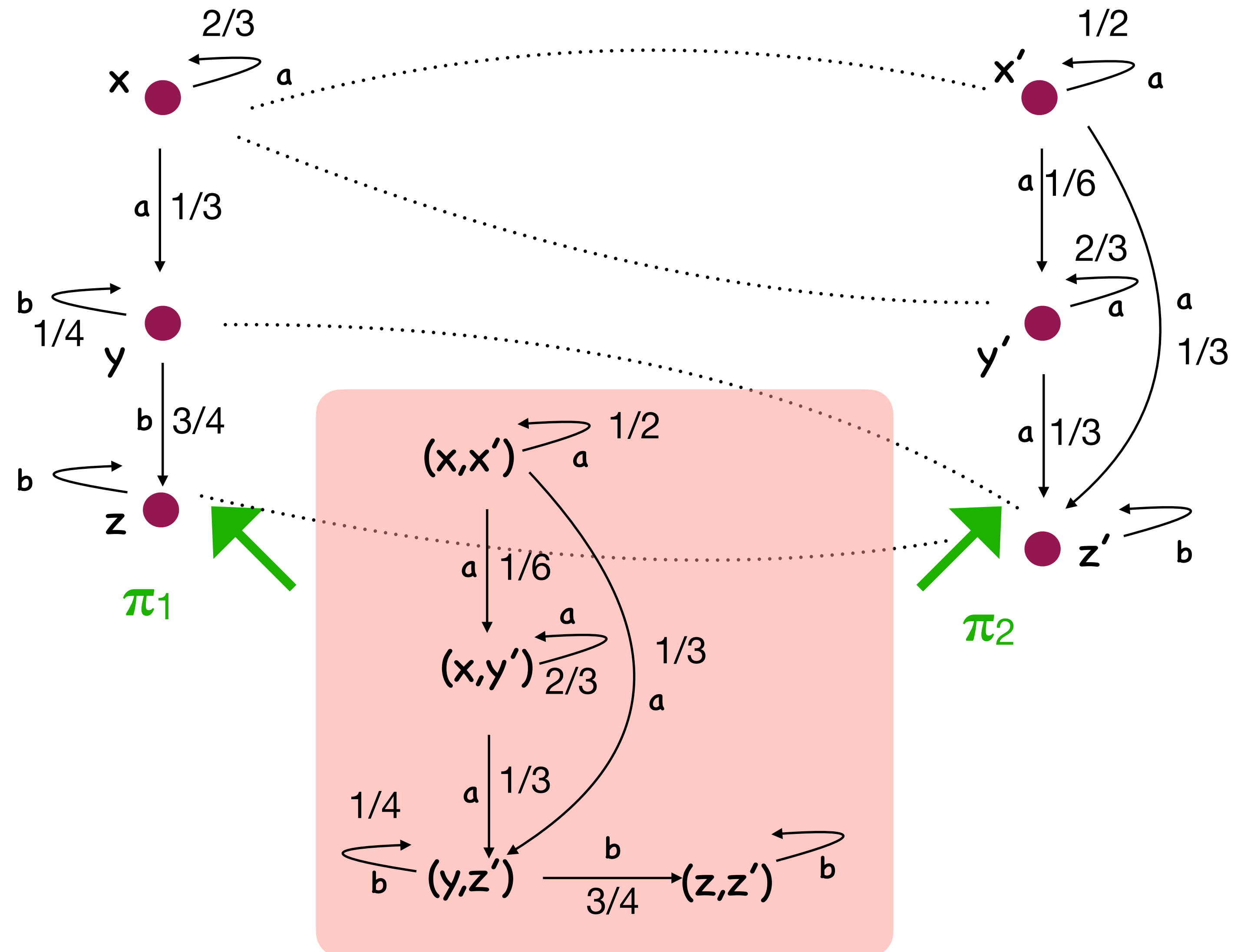


Bisimilarity of Probabilistic Transition Systems

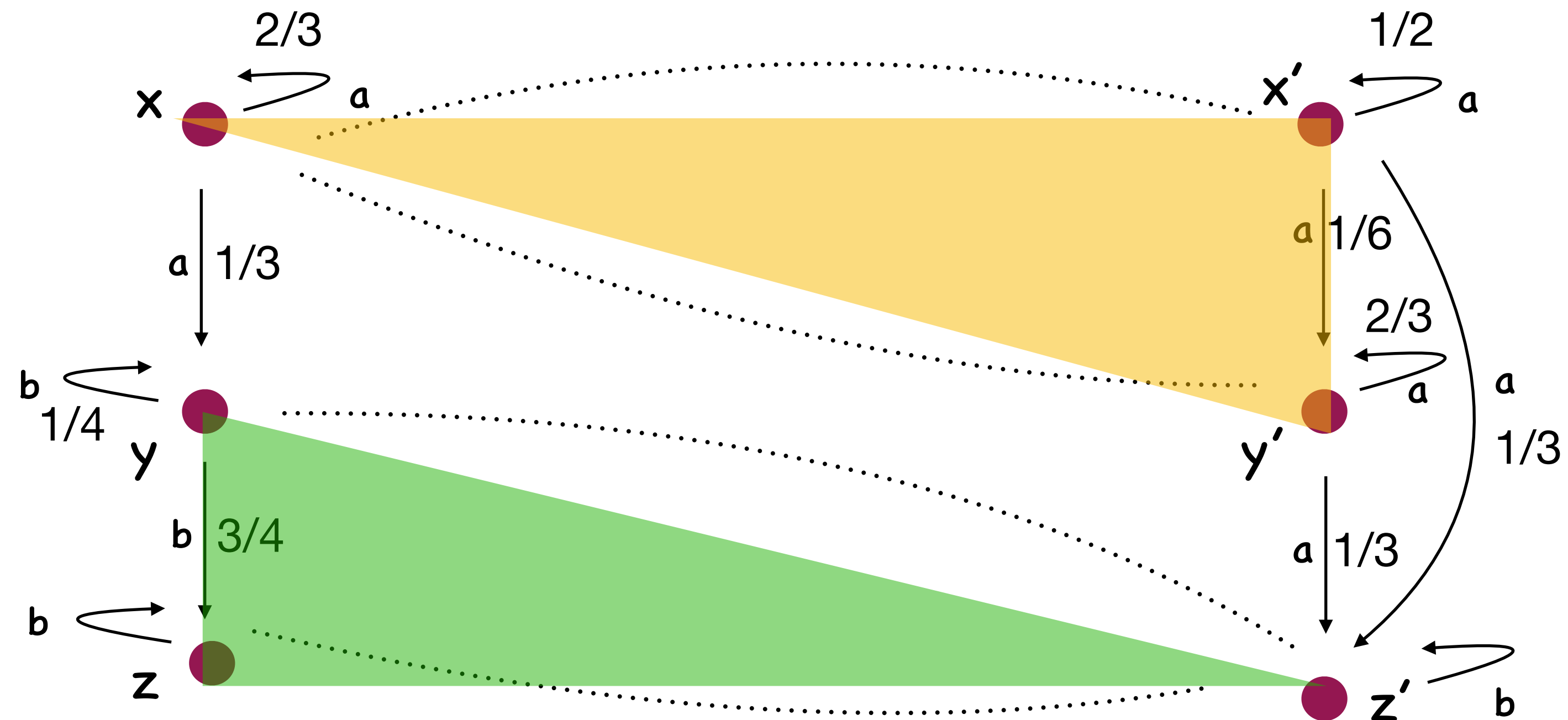
$$S \rightarrow (\mathcal{D} S)^A$$



Bisimilarity of Probabilistic Transition Systems



Bisimilarity of Probabilistic Transition Systems

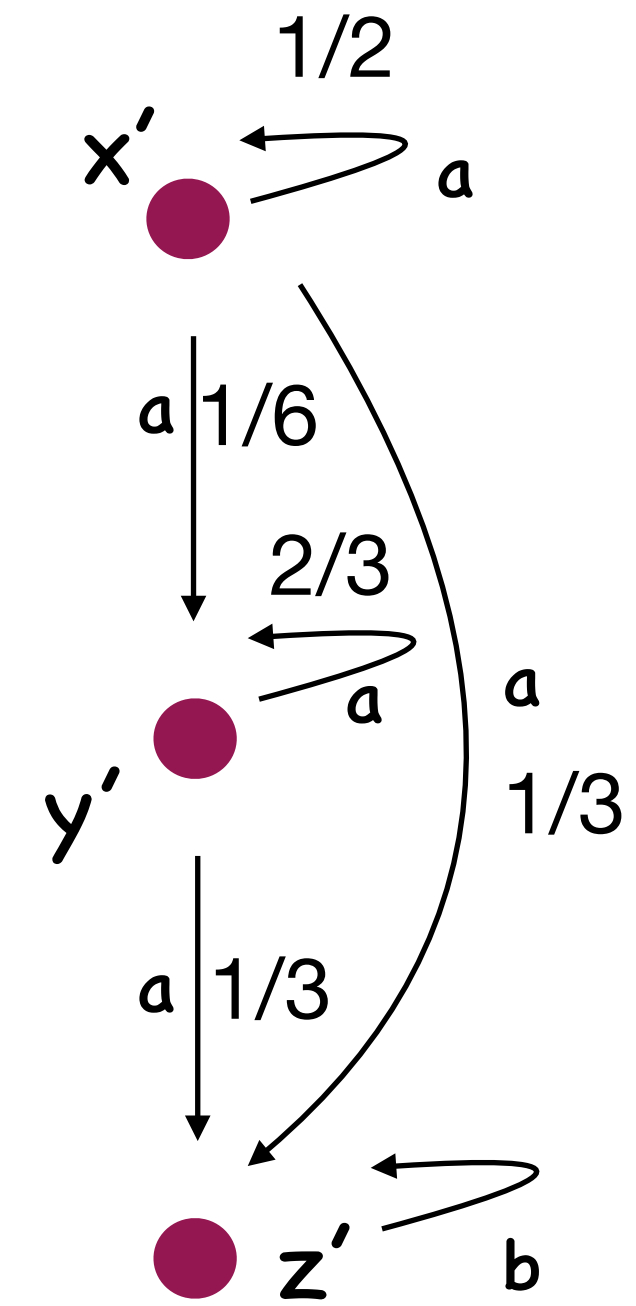
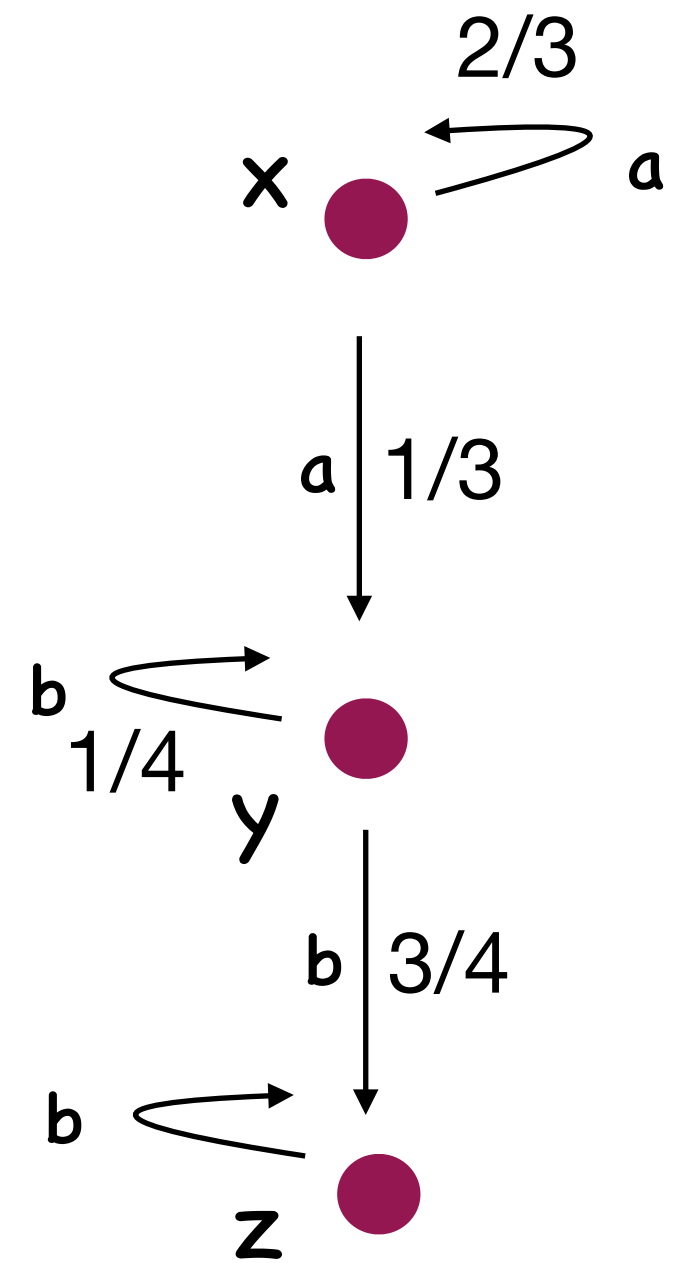


$$S \rightarrow (\mathcal{D} S)^A$$

A (probabilistic) bisimulation is an equivalence relation on states preserving probabilities towards equivalence classes.

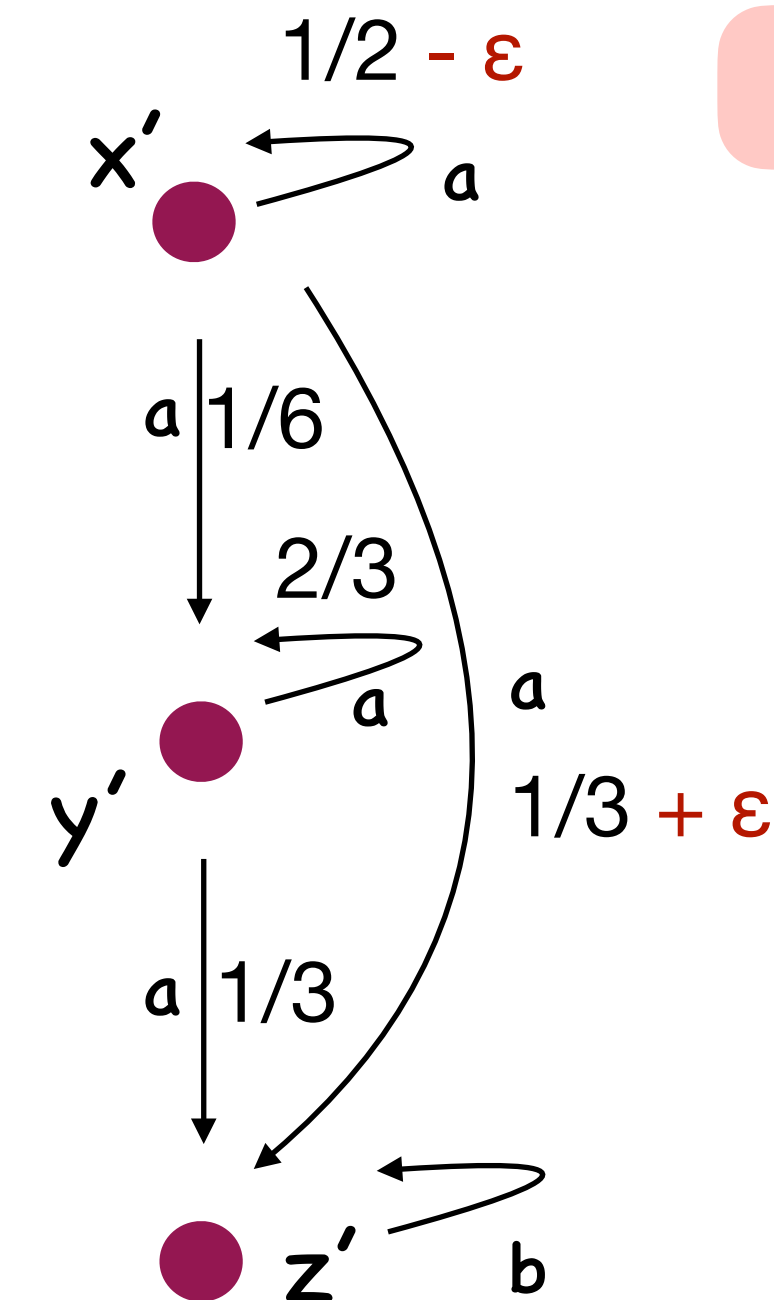
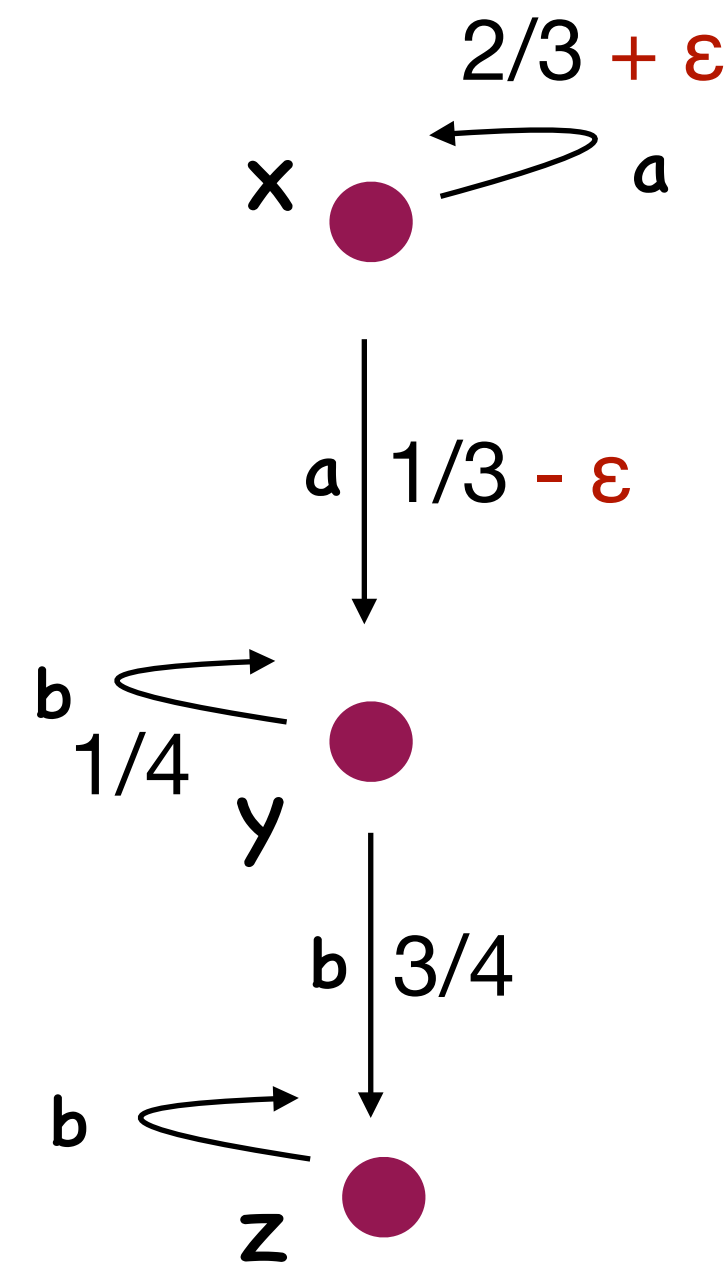
Larsen & Skou 1991

Sometimes bisimilarity is too strong / not robust



$$S \rightarrow (\mathcal{D} S)^A$$

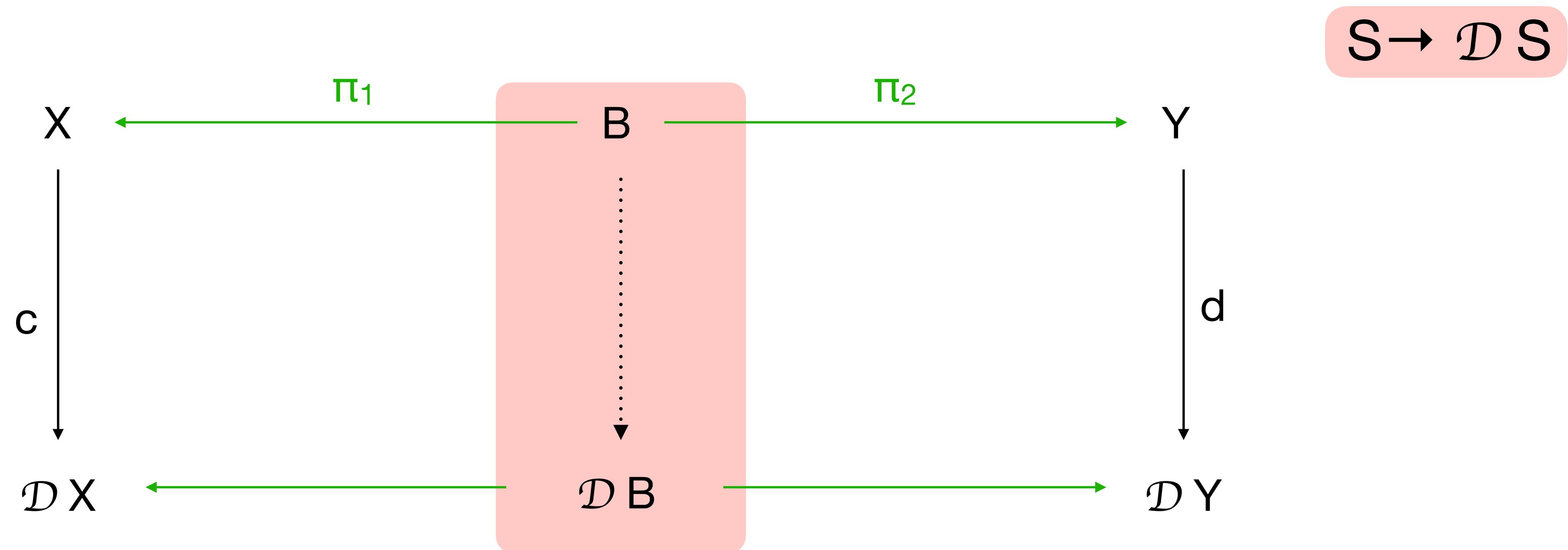
Sometimes bisimilarity is too strong / not robust



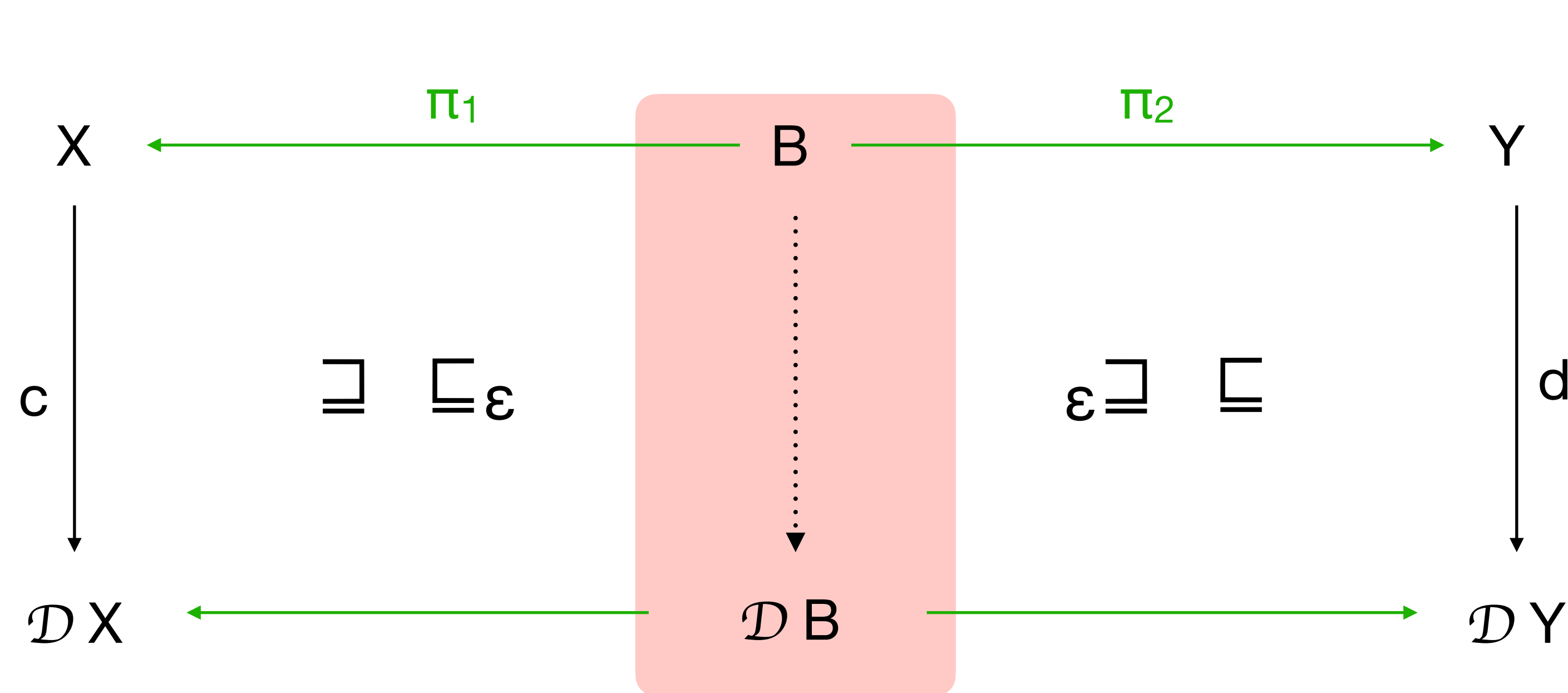
$$S \rightarrow (\mathcal{D} S)^A$$

Realistic situation, as probabilities are often approximations of real behaviour.

Recall bisimulation for PTS (one label)



ε -Bisimulation in a diagram

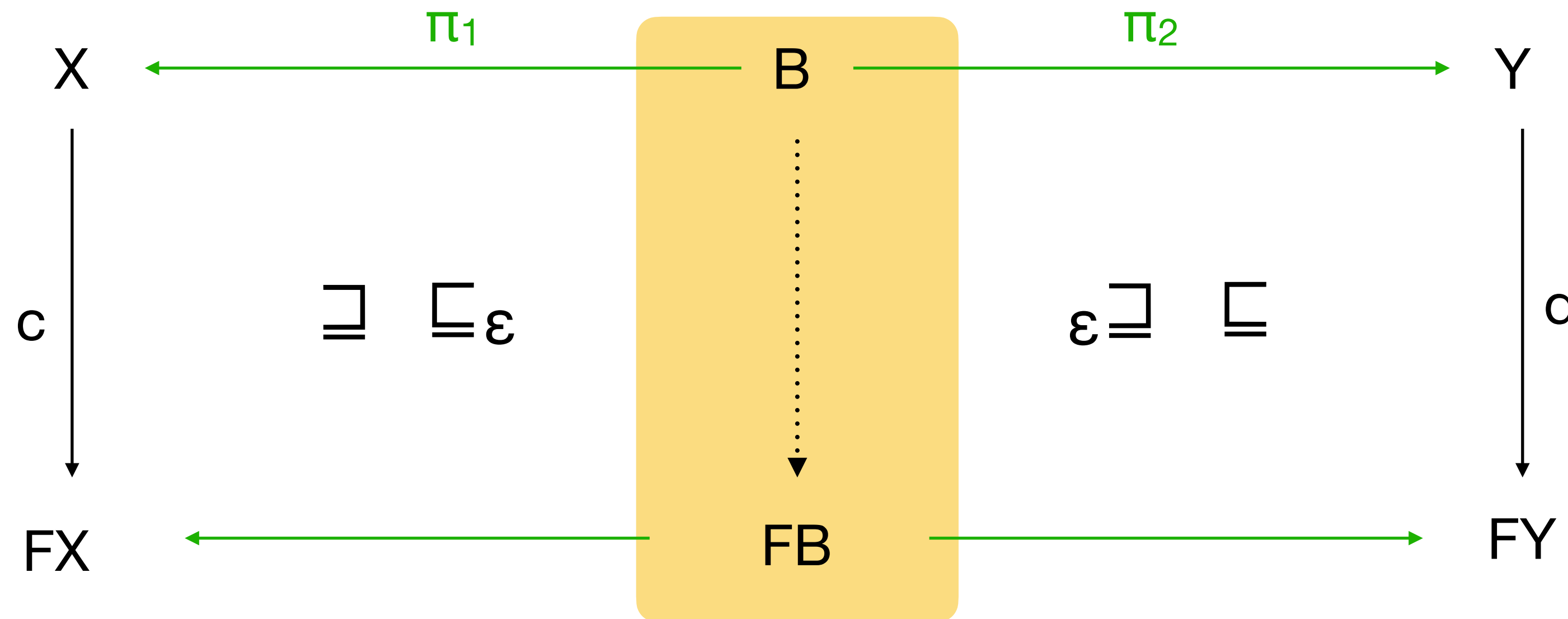


$$S \rightarrow \mathcal{D} S$$

with: $\varphi \subseteq_\varepsilon \psi$ iff for all $A \subseteq B$, $\varphi(A) \leq \psi(A) + \varepsilon$

ε -Bisimulation for F-coalgebras

$S \rightarrow FS$



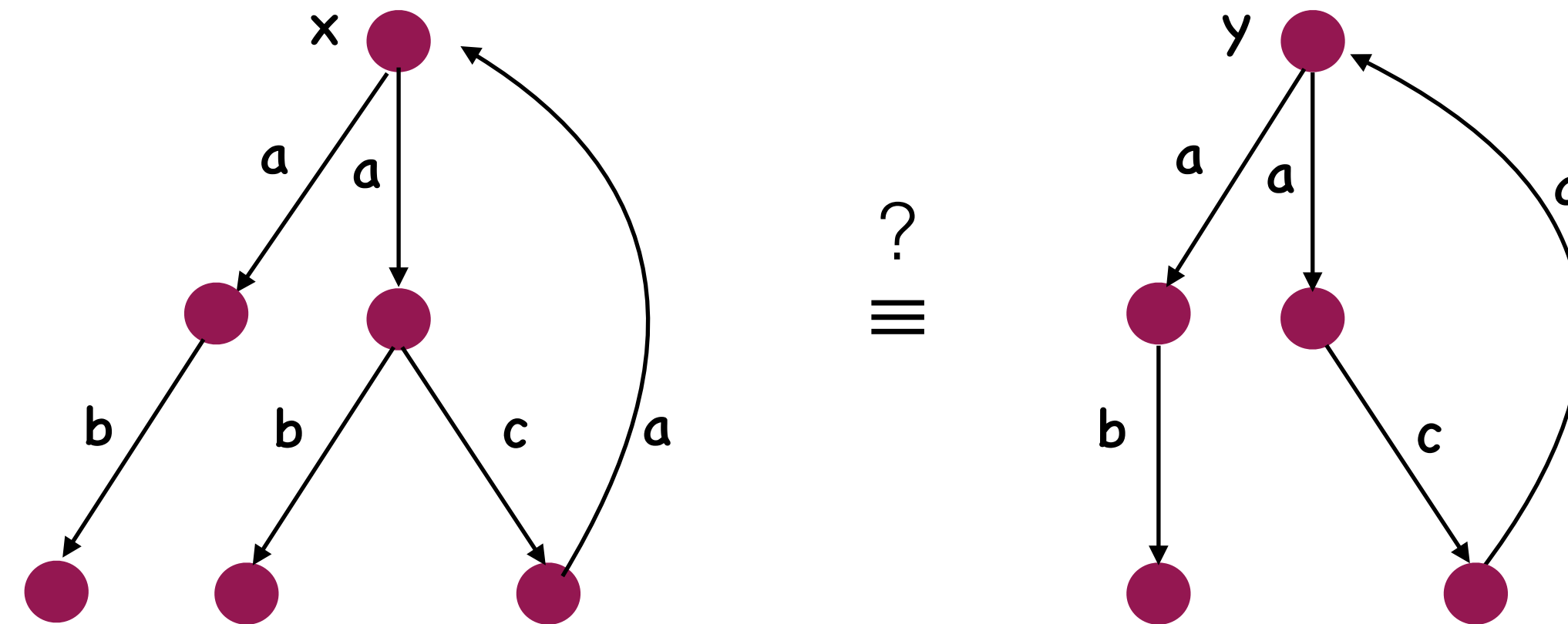
as long as F comes with a suitable order

this enables new definition of approximate bisimulations for other types of systems

II. Traces and Trace Semantics



Traces of Labelled Transition Systems



one can prove this using
equational reasoning
[Rabinovich'95] extending [Milner]

trace equivalent

perform the same traces
(=sequences of labels)

Traces of Labelled Transition Systems

An LTS is a coalgebra $X \rightarrow \mathcal{P}(A \times X)$
with a set of states X and a set of actions A .

We get a unique finite trace map $\text{tr}: X \rightarrow \mathcal{P}(A^*)$

by $aw \in \text{tr}(x) \Leftrightarrow \exists y \in X. x \xrightarrow{a} y \wedge w \in \text{tr}(y)$

(Finite) Trace equivalence is then defined as $x \equiv y \Leftrightarrow \text{tr}(x) = \text{tr}(y)$

Traces of Labelled Transition Systems

An LTS is a coalgebra $X \rightarrow \mathcal{P}(A \times X)$
with a set of states X and a set of actions A .

We get a unique **infinite** trace (stream) map $\text{str}: X \rightarrow \mathcal{P}(A^\omega)$

by $aw \in \text{str}(x) \Leftrightarrow \exists y \in X. x \xrightarrow{a} y \wedge w \in \text{str}(y)$

(**Infinite**) Trace (stream) equivalence is then defined as

$$x =_{\text{str}} y \Leftrightarrow \text{str}(x) = \text{str}(y)$$

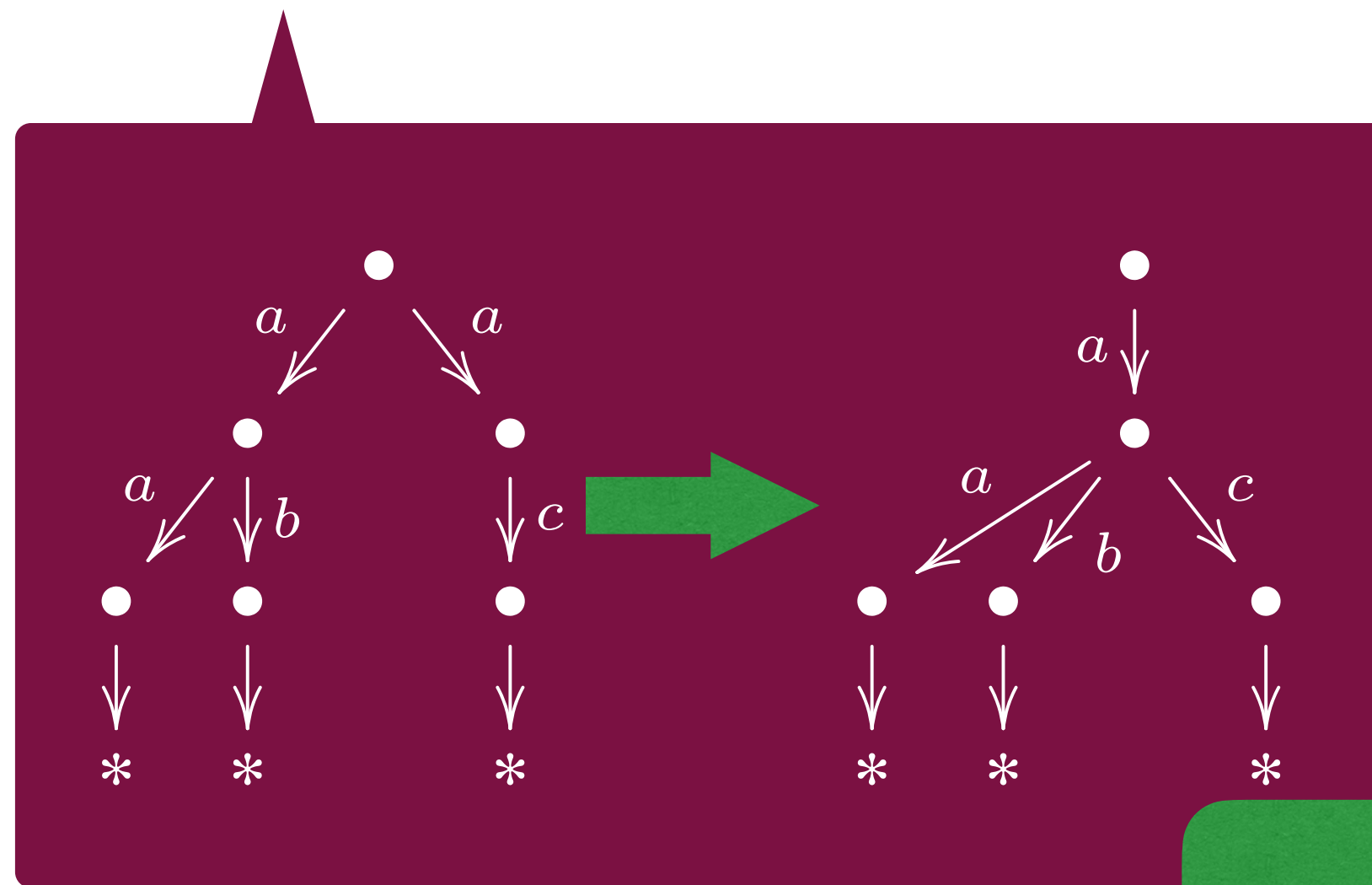
one can
combine
both for LTS

Trace semantics coalgebraically?

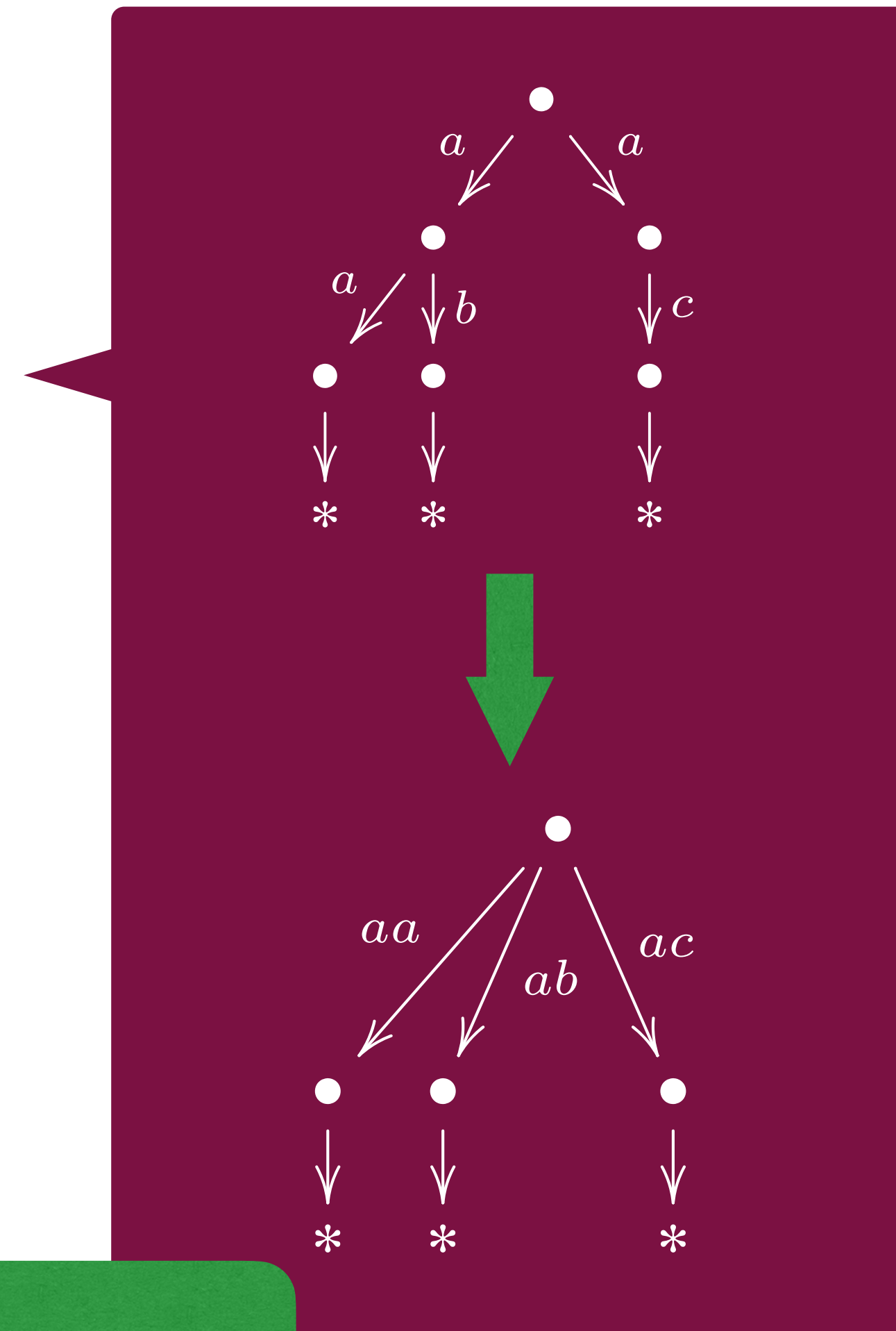
NFA / LTS

Two ideas:

- (1) unfold branching + transitions on words
- (2) trace = bisimilarity after determinisation



Monads !





Monads

Monads are functors that can be iterated.

A monad M is a functor that comes together with two natural transformations

$$\mu: MM \Rightarrow M \quad \eta: Id \Rightarrow M$$

Monads are
generalisation of
monoids

and an abstract view
on algebras

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

μ - union, η - singleton

$M = \mathcal{D}$
for probability

(Sub)Distributions

$M = \underline{\mathcal{PD}}$
for nondeterminism
and probability

Convex (n.e., f.g.)
subsets of distributions



Eilenberg-Moore Algebras

abstractly

$\mathcal{EM}(M)$

- objects

$$\begin{array}{c} MA \\ \downarrow a \\ A \end{array}$$

satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta} & MA \\ & \searrow a & \downarrow a \\ & & A \end{array}$$

$$\begin{array}{ccc} MMA & \xrightarrow{\mu} & MA \\ Ma \downarrow & & \downarrow a \\ MA & \xrightarrow{a} & A \end{array}$$

- morphisms

$$\begin{array}{c} MA \\ \downarrow a \\ A \end{array} \xrightarrow{h} \begin{array}{c} MB \\ \downarrow b \\ B \end{array}$$

$$\begin{array}{ccc} MA & \xrightarrow{Mh} & MB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{h} & B \end{array}$$

Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

free algebras of a monad
 M

(2) modelling in an Eilenberg-Moore category

algebras of a monad M

(1) and (2) are related

Hasuo, Jacobs, S.
LMCS '07

Silva, Bonchi,
Bonsangue, Rutten
FSTTCS'10

Jacobs, Silva, S.
JCSS'15

Trace semantics
is about iteration!

Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

$$X \rightarrow MFX \rightarrow MF\textcolor{red}{M}FX \rightarrow MMFFX \rightarrow MFFX \rightarrow \dots$$

(2) modelling in an Eilenberg-Moore category

$$X \rightarrow FMX, \quad MX \rightarrow \textcolor{red}{M}FMX \rightarrow FMMX \rightarrow FMX$$

Hasuo, Jacobs, S.
LMCS '07

Silva, Bonchi,
Bonsangue, Rutten
FSTTCS'10

Trace semantics
is about iteration!

The Kleisli approach

generative

MF- coalgebras

unfold branching + transitions on words

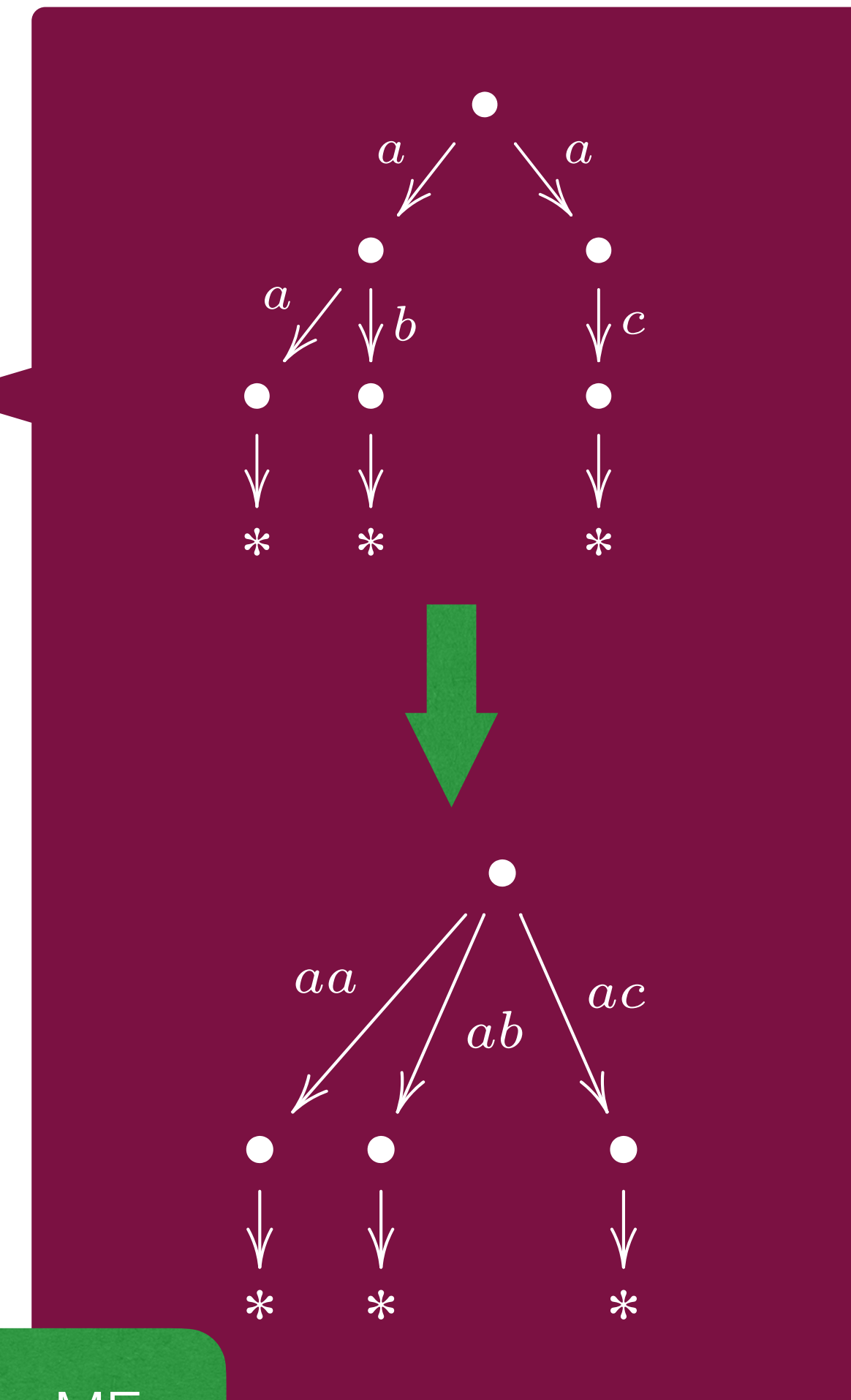
$$X \xrightarrow{c} \underline{MFX} \xrightarrow{MFc} \underline{MF} \underline{MFX} \xrightarrow{\text{d.l.}} \underline{MMFFX} \xrightarrow{\mu} \underline{MFFX}$$

trace by coinduction in $KI(M)$

initial algebra
yields final coalgebra
in $KI(M)$

works under
certain
order-theoretic
conditions

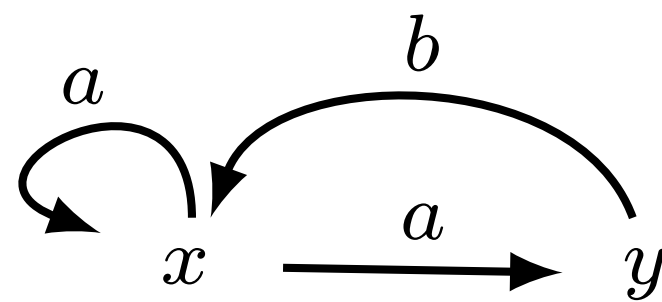
needed d.l. $FM \Rightarrow MF$



Coalgebra examples - generative

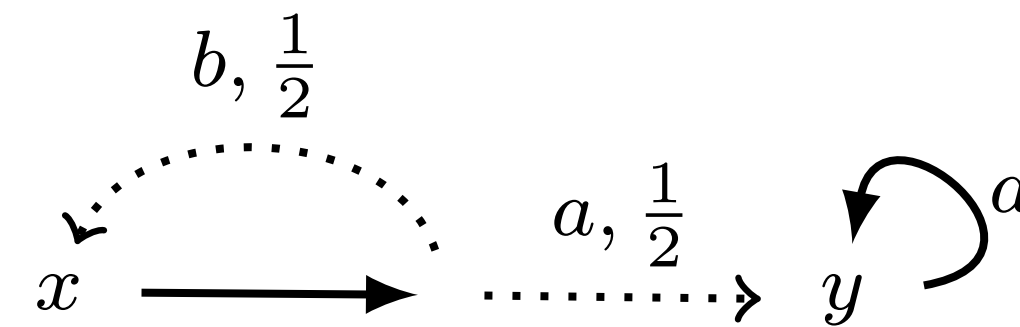
LTS

$$X \rightarrow \mathcal{P}(A \times X)$$



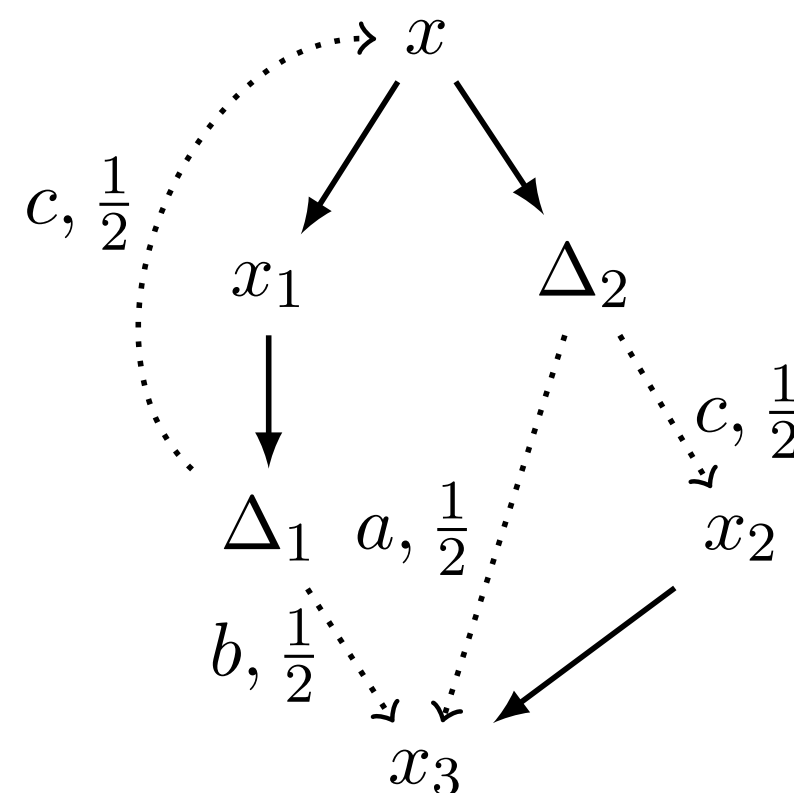
GPTS

$$X \rightarrow \mathcal{D}(A \times X)$$



GNPLTS

$$X \rightarrow \mathcal{PD}(A \times X)$$

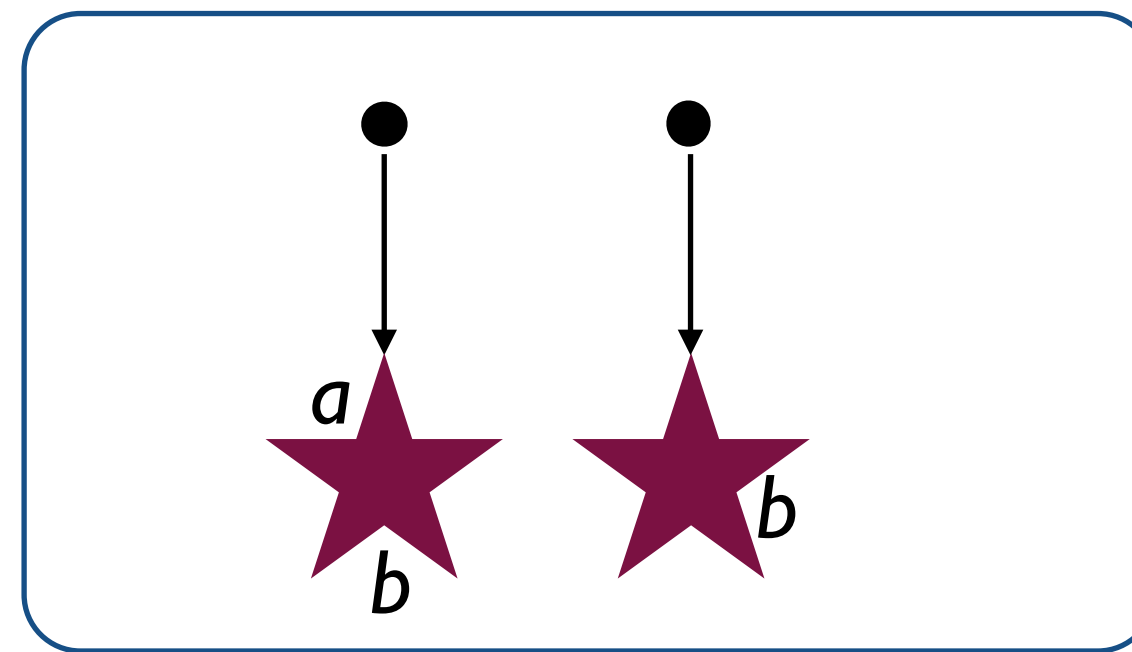


generative
system with
nondeterminism
and
probability

In general

Generative Systems

$$X \rightarrow M(A \times X)$$



with M-effects

for a monad M

providing
algebraic
effects

The Eilenberg-Moore approach

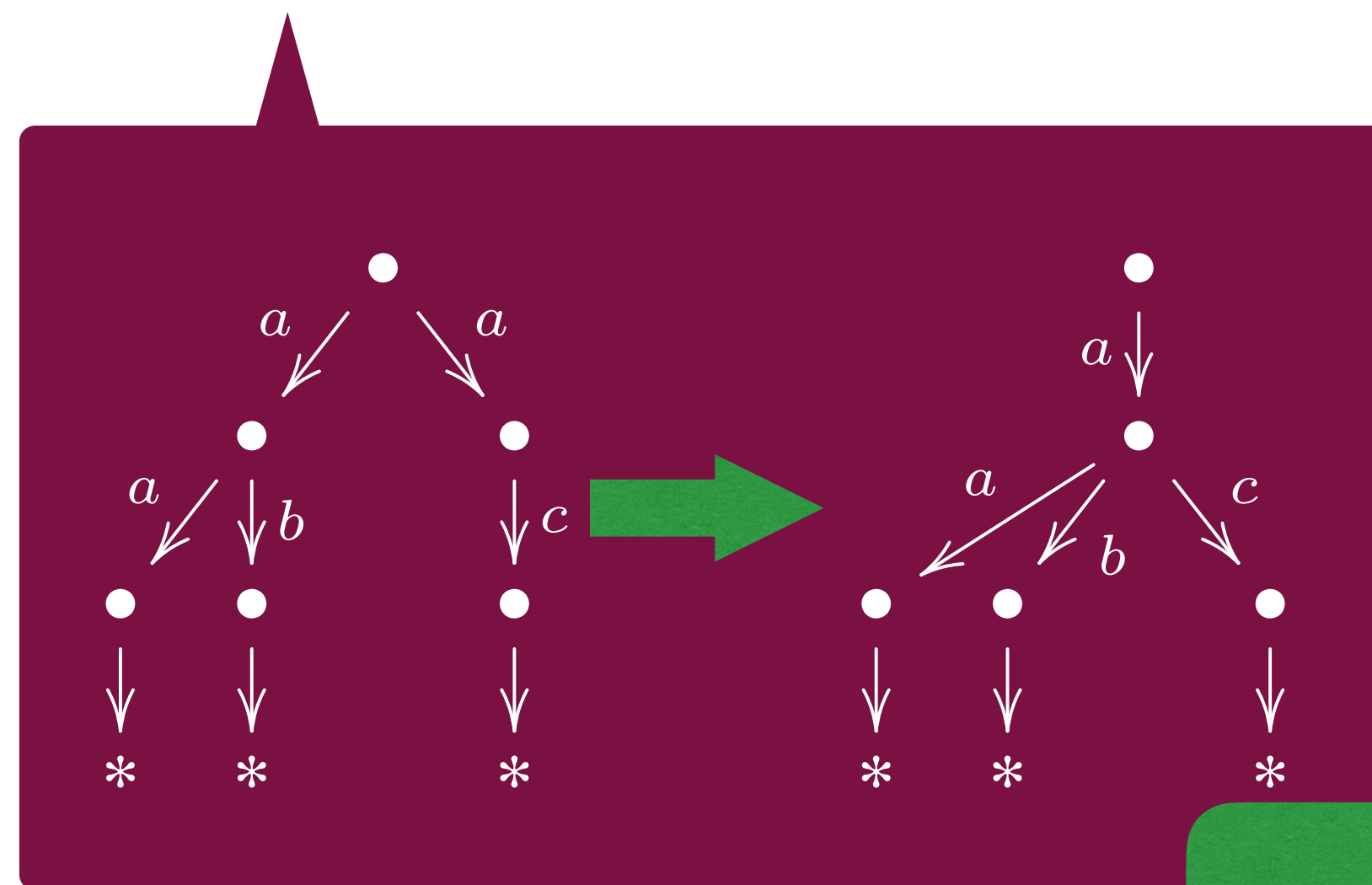
reactive

FM-coalgebras

Silva, Bonchi,
Bonsangue, Rutten
FSTTCS'10

trace = bisimilarity after determinisation

$$\underline{MX} \xrightarrow{M_c} \underline{MFX} \xrightarrow{\text{d.l.}} \underline{FMMX} \xrightarrow{\mu} \underline{FMX}$$



F final coalgebra
yields final coalgebra
in EM(M)

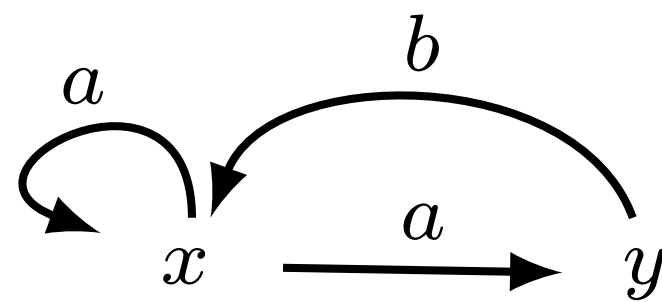
trace by
coinduction in
EM(M)

needed d.l. MF $\Rightarrow$ FM

Coalgebra examples - reactive

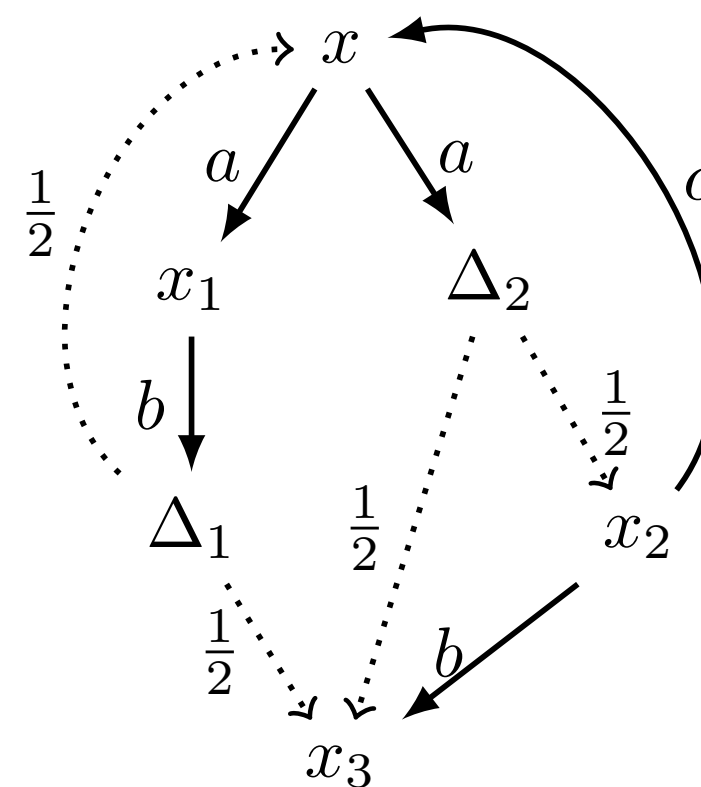
LTS

$$X \rightarrow (\mathcal{P}X)^A$$



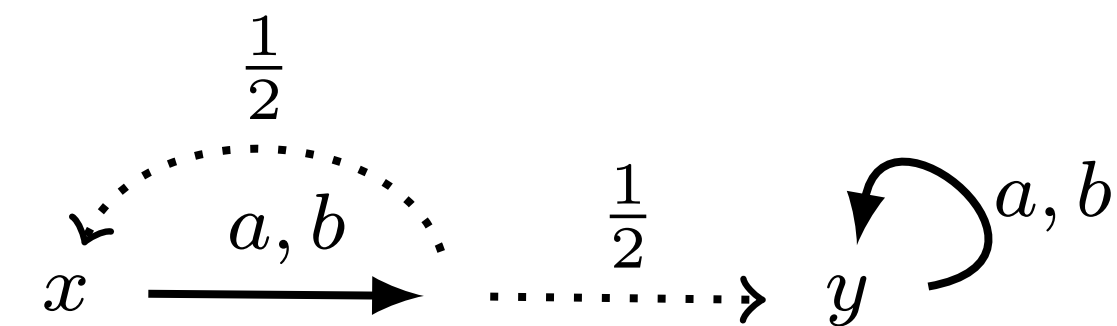
RNPLTS

$$X \rightarrow (\mathcal{P}\mathcal{D}X)^A$$



RPTS

$$X \rightarrow (\mathcal{D}X)^A$$

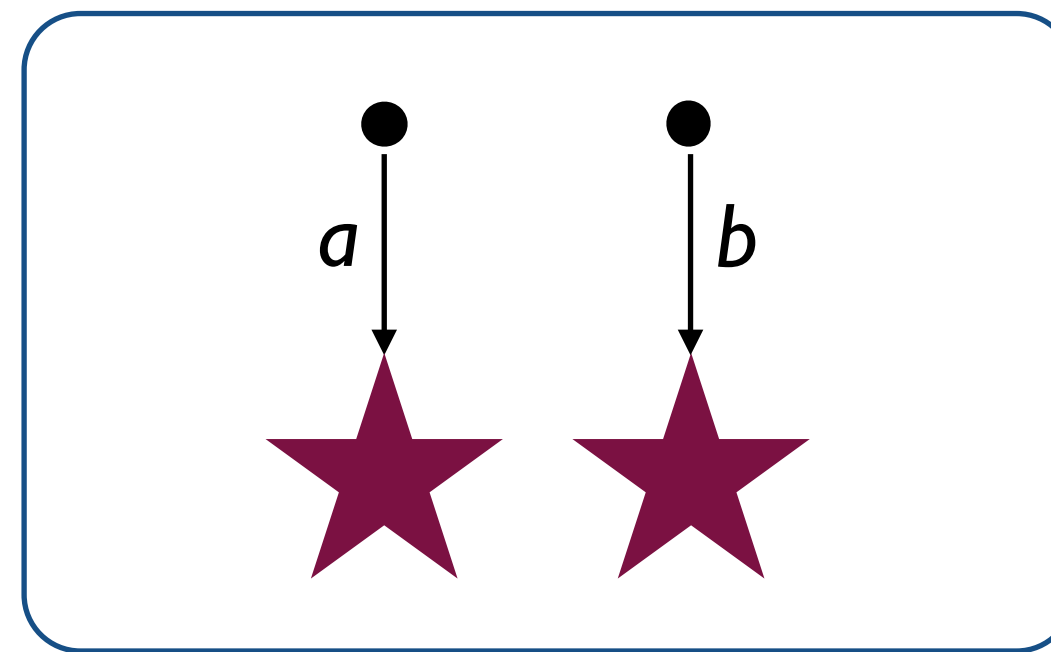


reactive
system with
nondeterminism
and
probability

In general

Reactive Systems

$$X \rightarrow (MX)^A$$



with M-effects

for a monad M

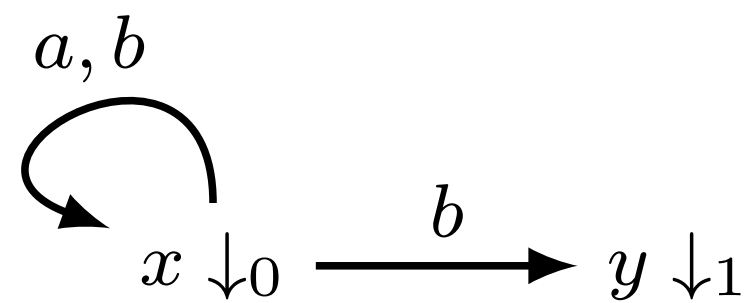
providing
algebraic
effects

we write $x \xrightarrow{a} t_x$

Automata

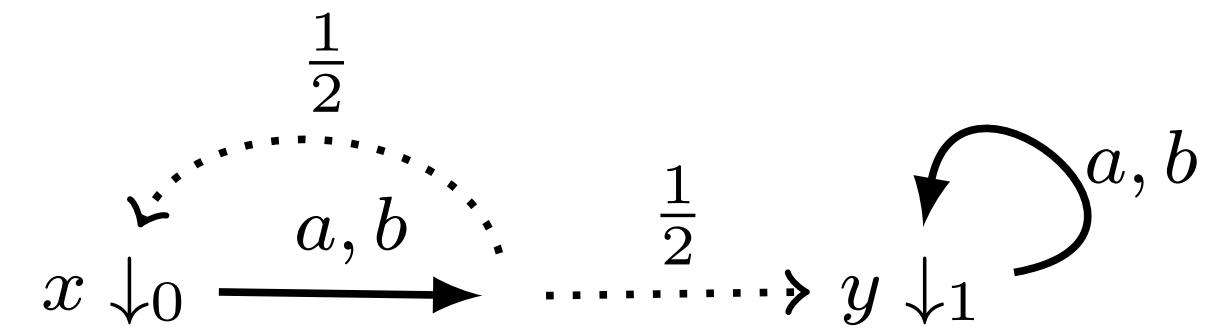
NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$



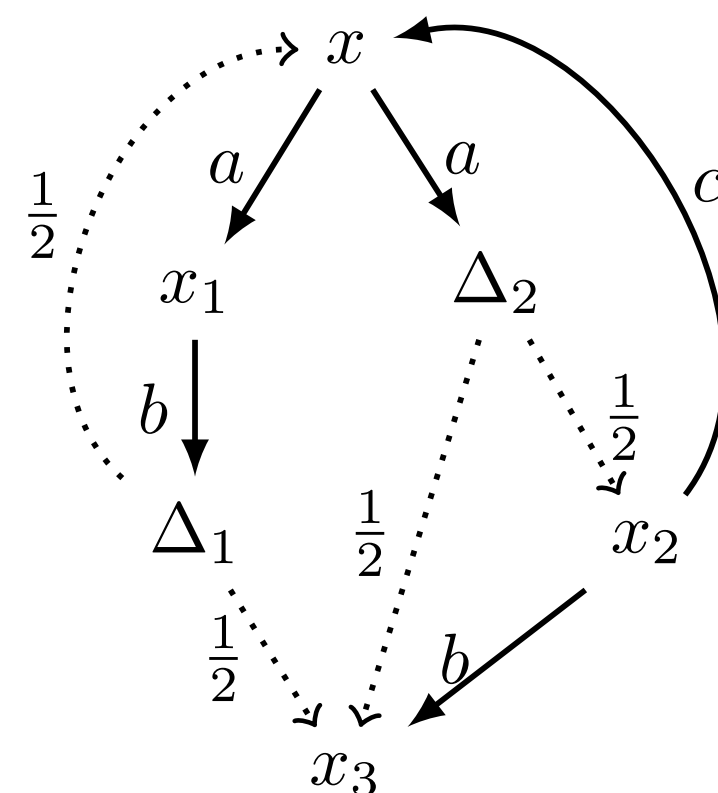
Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}X)^A$$



Simple NPA

$$X \rightarrow ? \times (\mathcal{P}\mathcal{D}X)^A$$



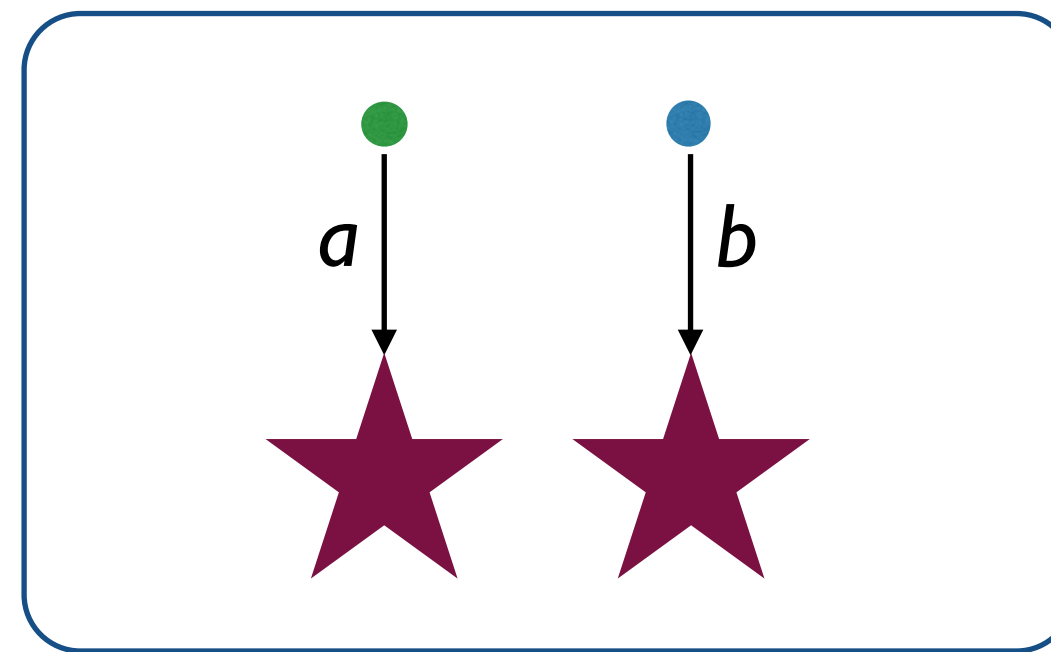
automata =
systems
with
observations

In general

Automata

$$X \rightarrow O \times (MX)^A$$

with
observations
in O



and M-effects

for a monad M

providing
algebraic
effects

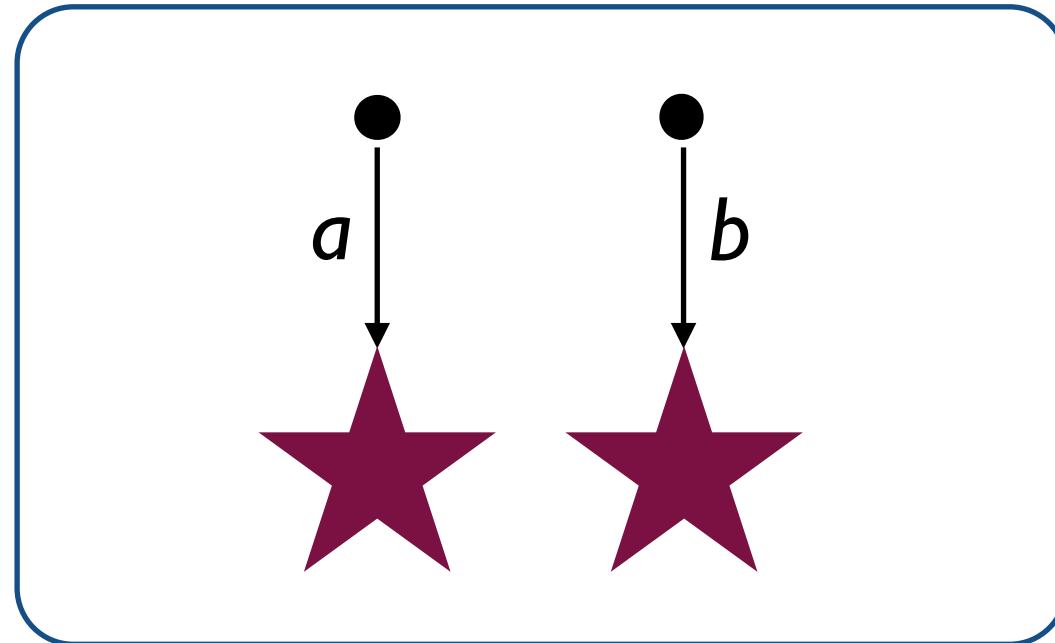
we write $x \downarrow o, x \xrightarrow{a} t_x$

$$tr: X \rightarrow O^{A^*}$$

Canonical observations

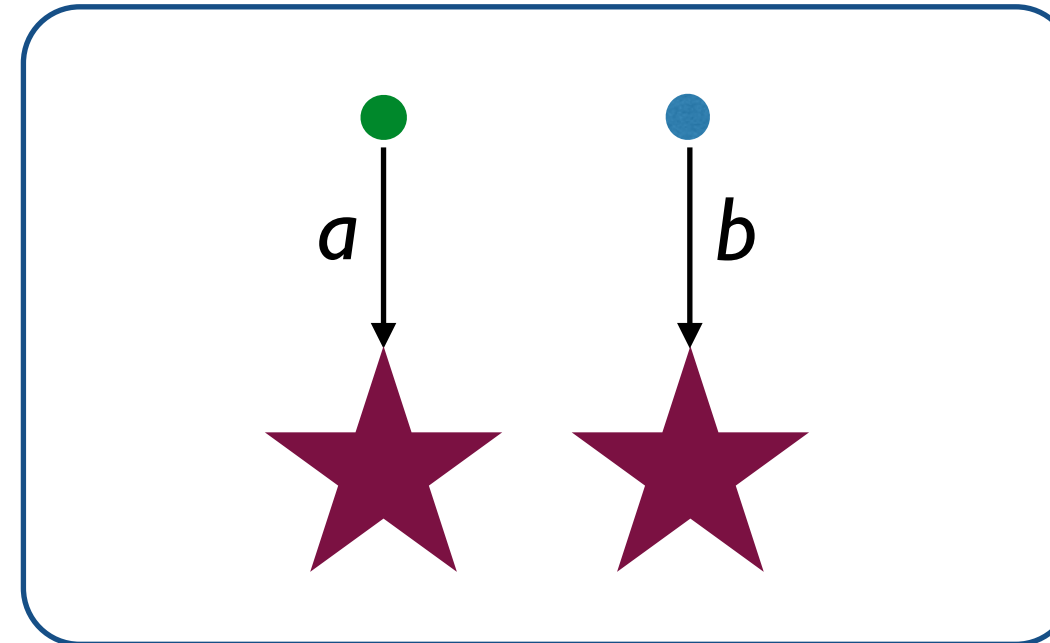
Reactive Systems

$$X \rightarrow (MX)^A$$



Automata

$$X \rightarrow O \times (MX)^A$$



Automata = Systems with observations in O

Systems = Automata with canonical observations

$$X \xrightarrow{!} 1 \xrightarrow{\eta} M1$$

In our examples

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$$X \rightarrow [0, 1] \times (\mathcal{D}X)^A$$

$M = \mathcal{D}$
for probability

(Sub)Distributions

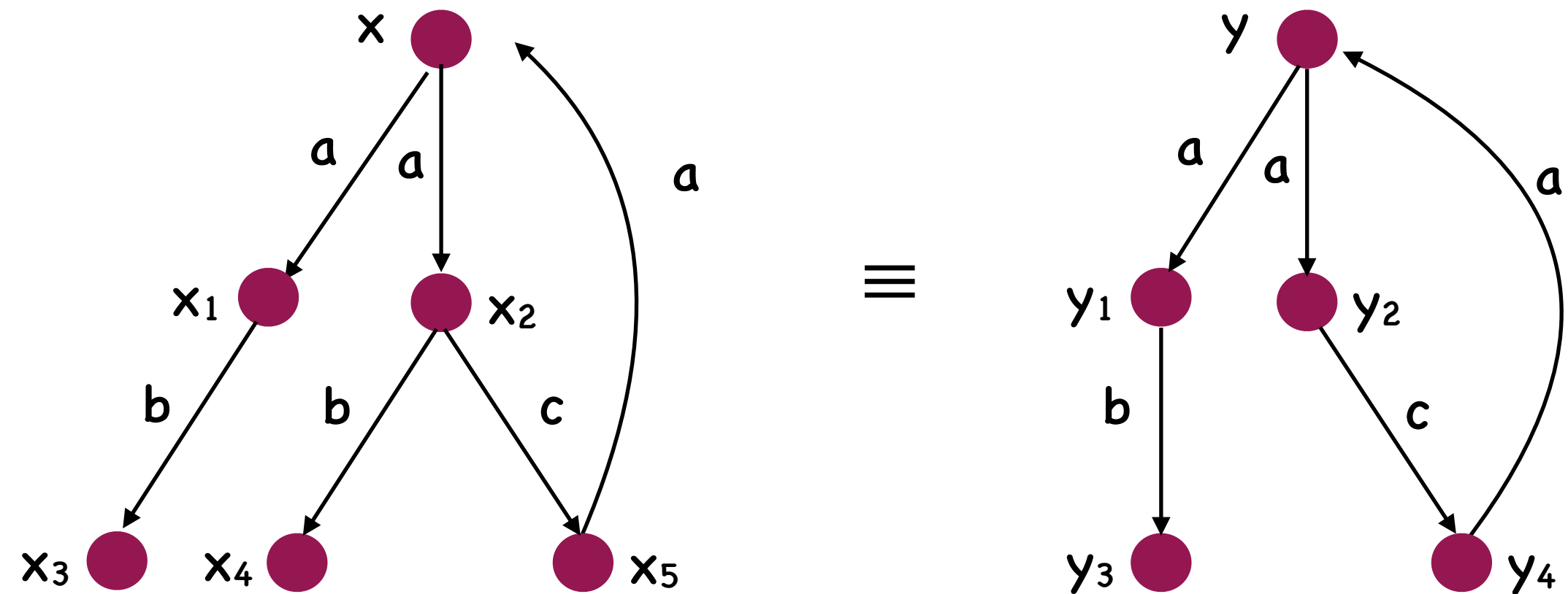
Simple PA

$$X \rightarrow ? \times (\underline{\mathcal{PD}}X)^A$$

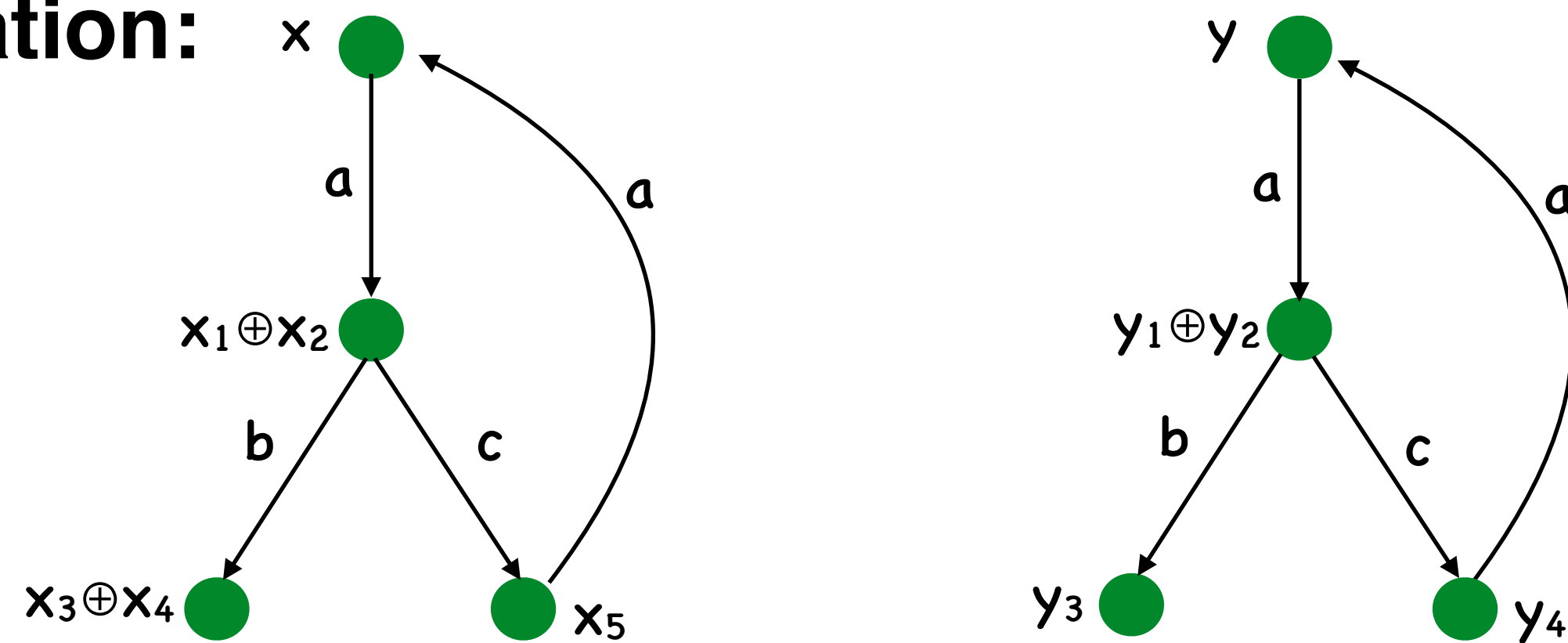
$M = \underline{\mathcal{PD}}$
for nondeterminism
and probability

Convex (n.e., f.g.)
subsets of distributions

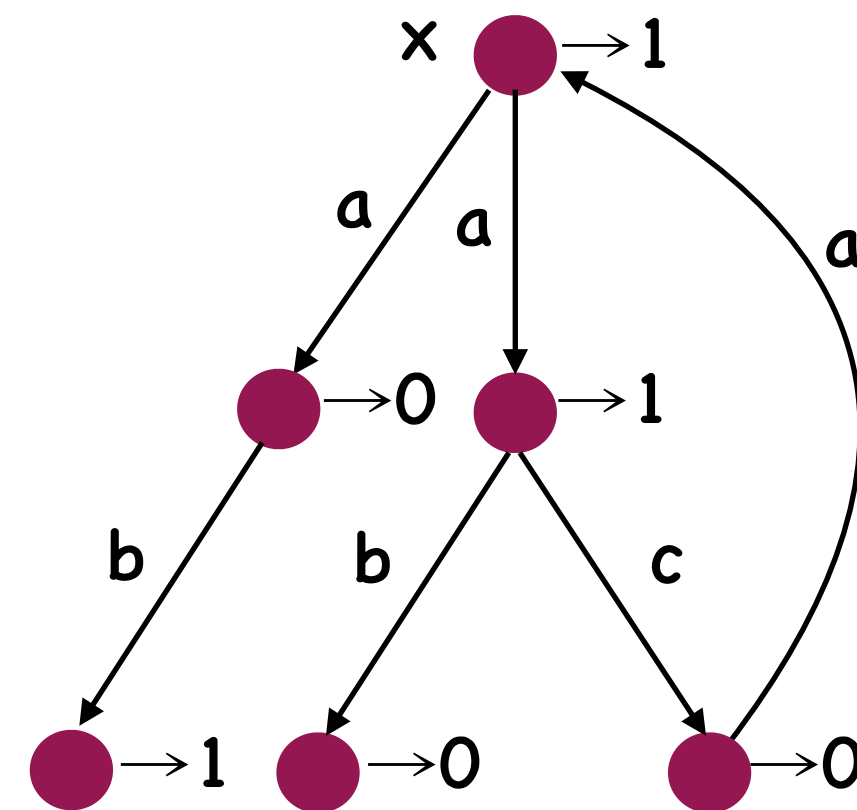
Traces of Labelled Transition Systems



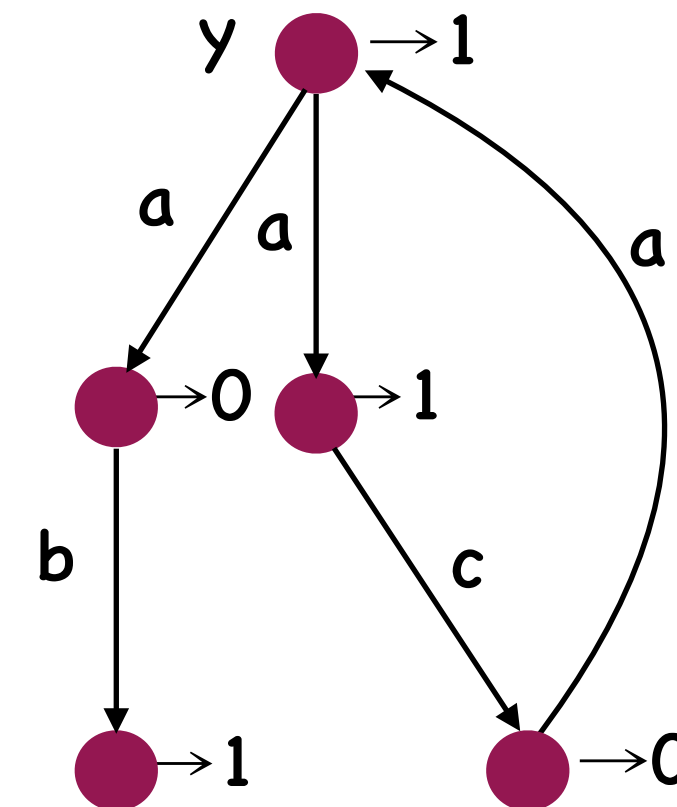
Bisimilar after determinization:



Traces of Nondeterministic Automata



?
≡



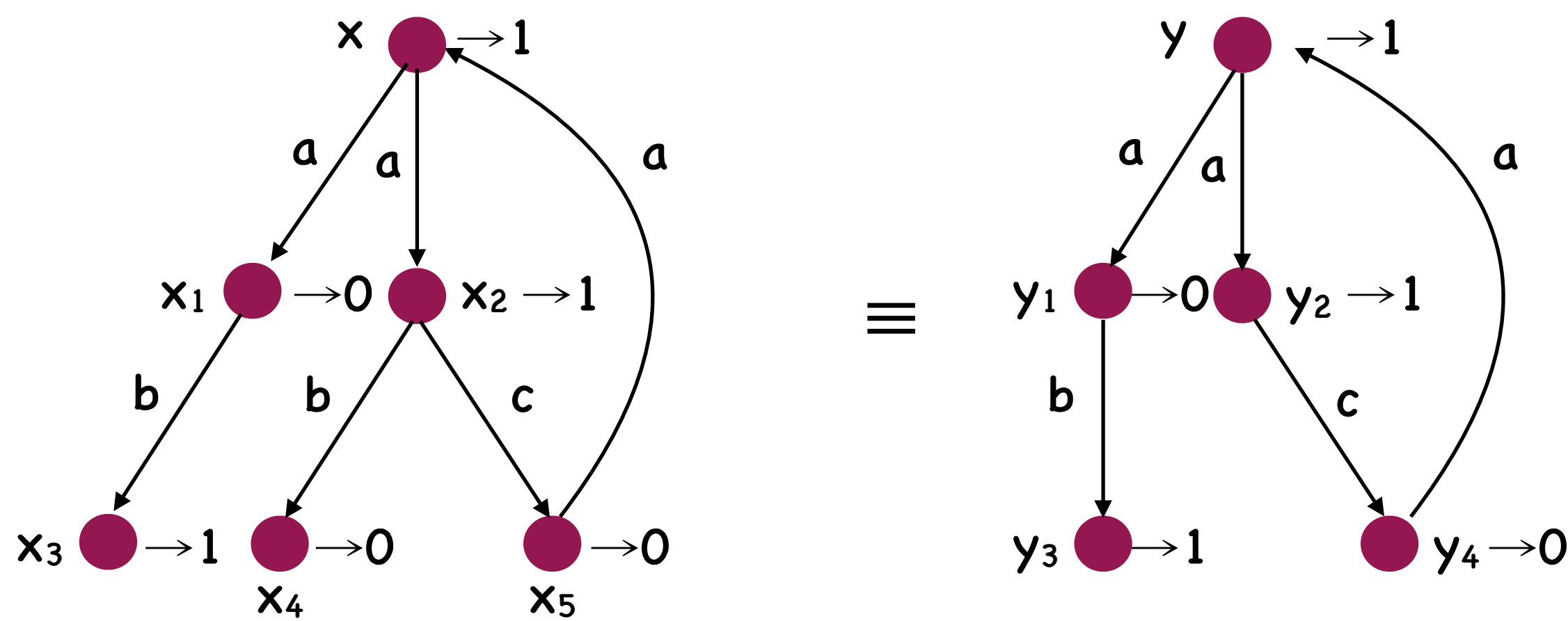
Free JSLB
{1}

$\oplus$	0	1
0	0	1
1	1	1

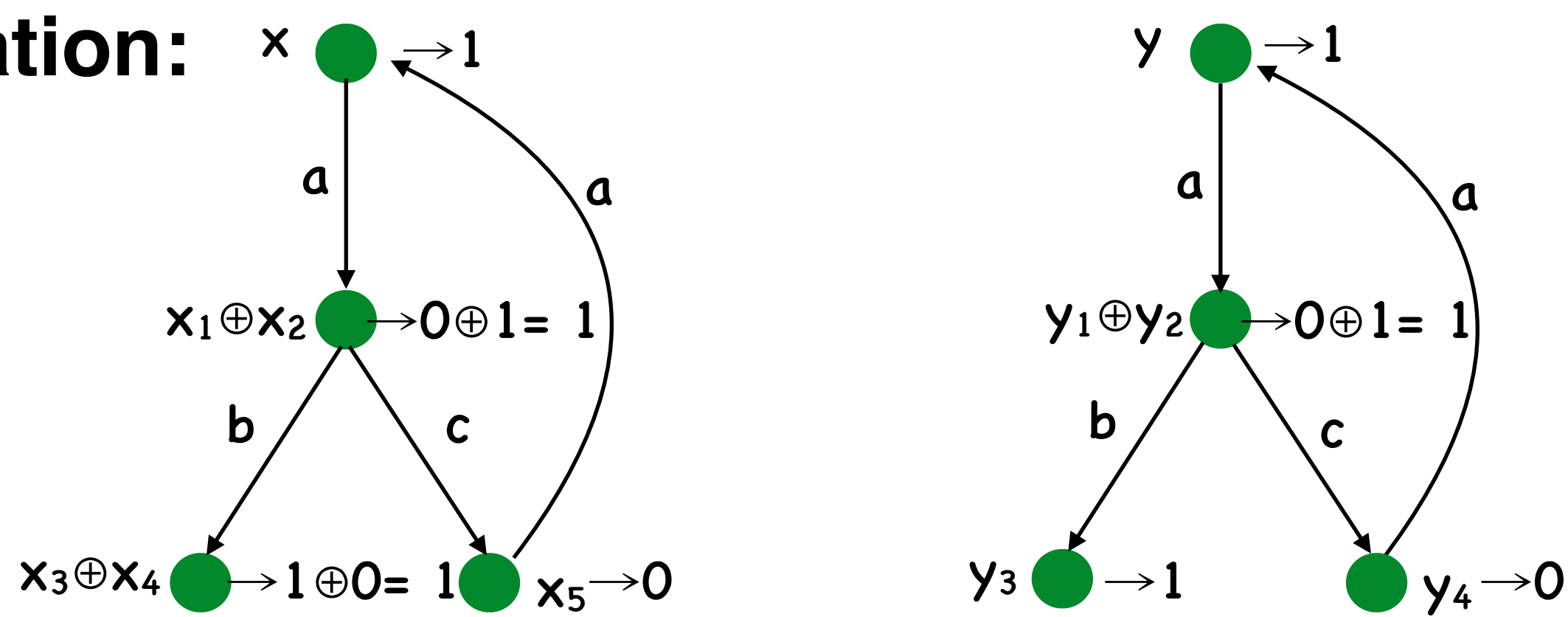
$$X \rightarrow \{0,1\} \times (\mathcal{P}X)^A$$

$$X \rightarrow \{0,1\} \times (\text{Free-JSLB } X)^A$$

Traces of Nondeterministic Automata



Bisimilar after determinization:



Free JSLB
{1}

$\oplus$	0	1
0	0	1
1	1	1

The Algebraic Theory of (Join) Semilattices with Bottom

$$s, t ::= \star, s \oplus t, x \in X$$

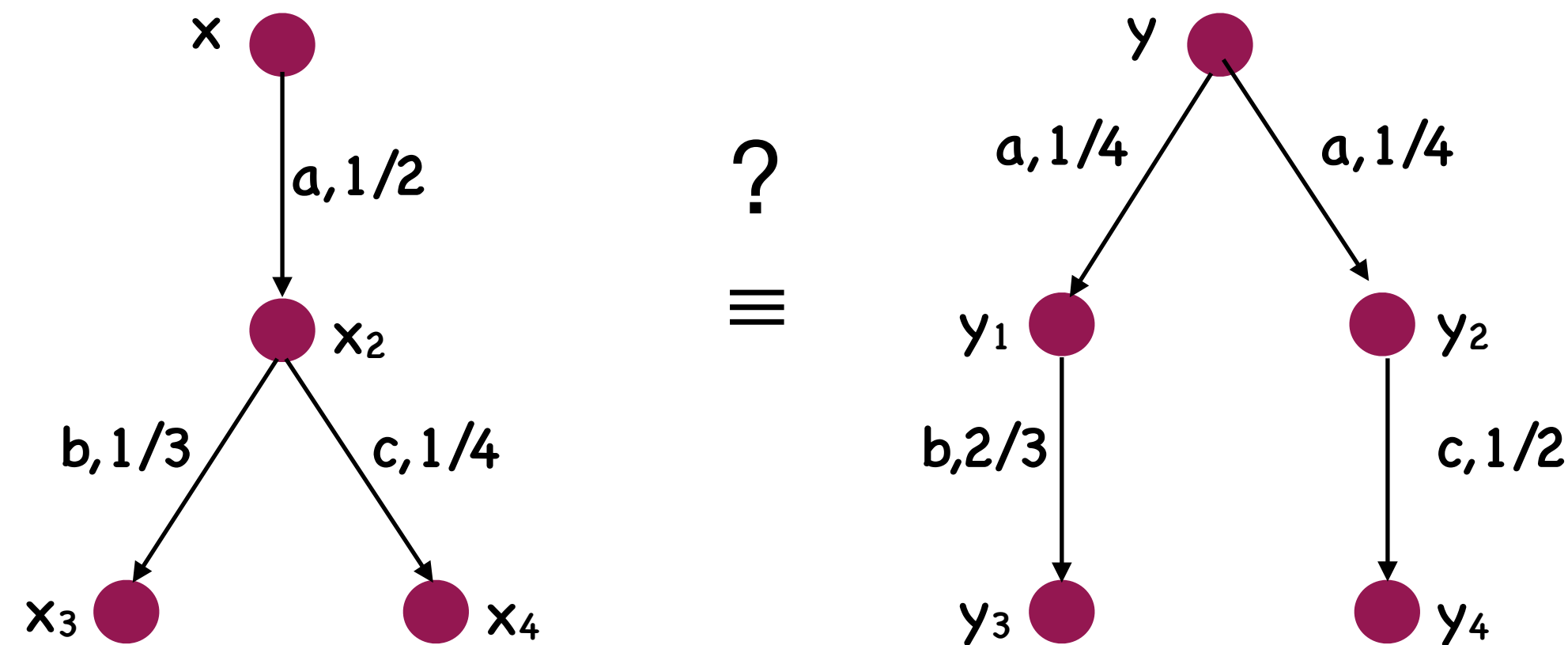
$$\begin{array}{ccc} (x \oplus y) \oplus z & \stackrel{(A)}{=} & x \oplus (y \oplus z) \\ x \oplus y & \stackrel{(C)}{=} & y \oplus x \\ x \oplus x & \stackrel{(I)}{=} & x \\ x \oplus \star & \stackrel{(B)}{=} & x \end{array}$$

The set of terms quotiented by these axioms is isomorphic to $\mathcal{P}X$

this theory is a presentation for the powerset monad

finite

Traces of Probabilistic Transition Systems



one can prove this using
equational reasoning
[Silva&S.'11]

trace equivalent

perform the same traces
(=sequences of labels)
with the same probability

$$X \rightarrow (\mathcal{D}X)^A$$

sum over all paths,
product along a path

$$X \rightarrow (\text{Free-PCA } X)^A$$

Traces of Probabilistic Transition Systems

A PTS is a coalgebra $c: X \rightarrow \mathcal{D}(A \times X)$

with a set of states X and a set of actions A .

We get a unique finite trace map $\text{tr}: X \rightarrow \mathcal{D}(A^*)$

by
$$\text{tr}(x)(aw) = \sum_y c(x)(a, y) \cdot \text{tr}(y)(w)$$

(Finite) Trace equivalence is then defined as $x \equiv y \Leftrightarrow \text{tr}(x) = \text{tr}(y)$

Traces of Probabilistic Transition Systems

A PTS is a coalgebra $c: X \rightarrow \mathcal{D}(A \times X)$

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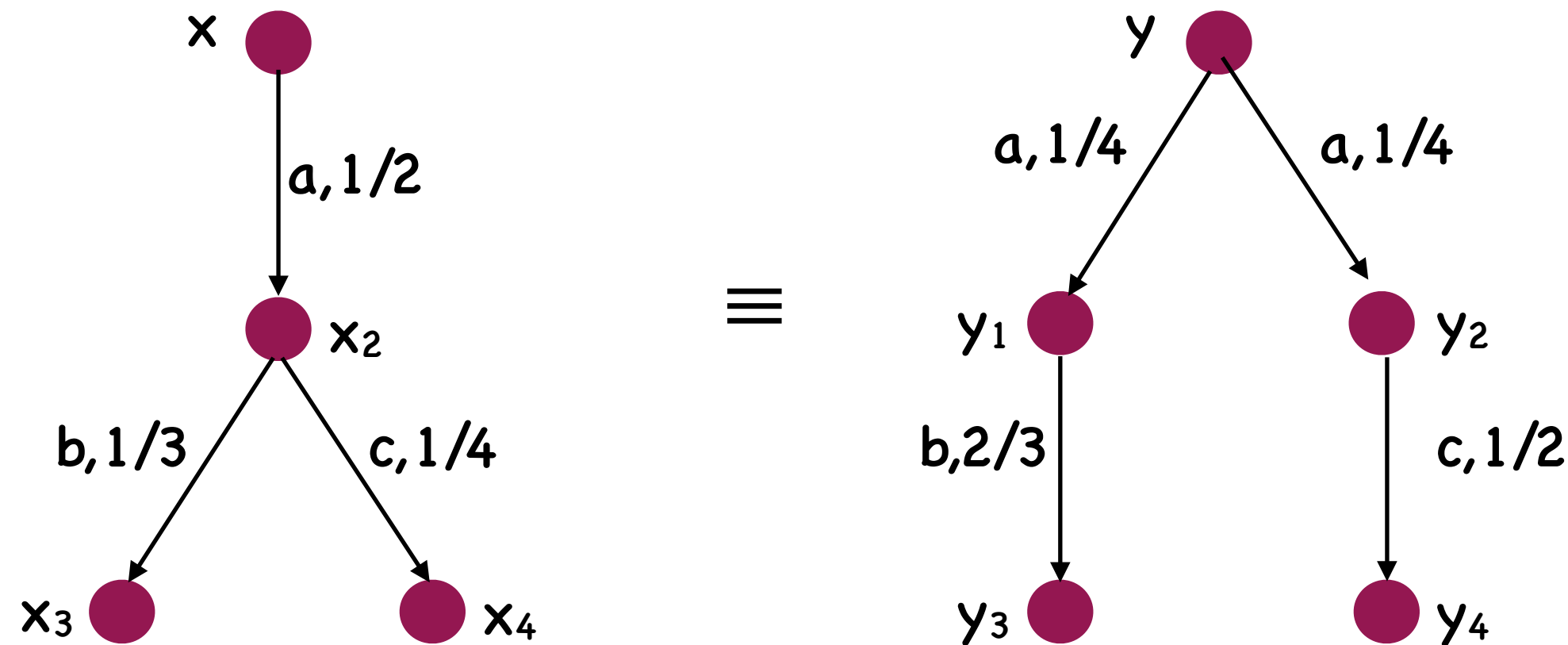
We get a unique **infinite** trace (stream) map $\text{str}: X \rightarrow \text{Prob}(A^\omega)$

by
$$\text{str}(x)(B_{aw}) = \sum_y c(x)(a, y) \cdot \text{str}(y)(B_w)$$

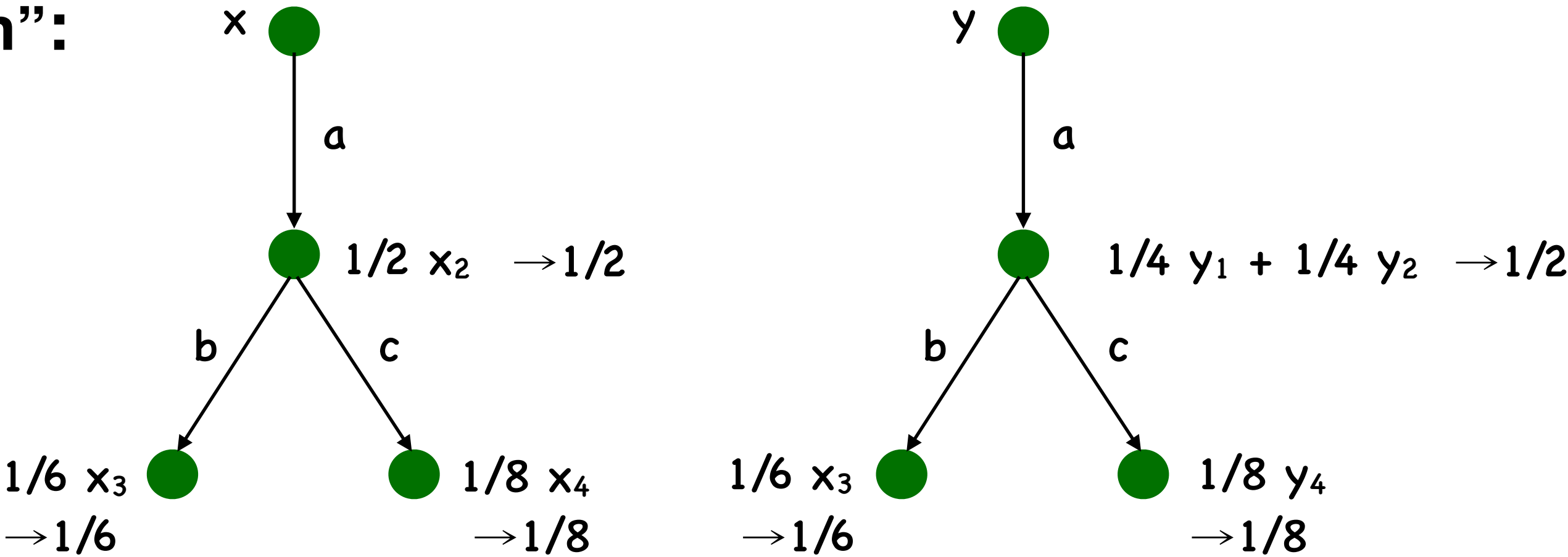
(Ininite) Trace (stream) equivalence is then defined as

$$x =_{\text{str}} y \Leftrightarrow \text{str}(x) = \text{str}(y)$$

Traces of Probabilistic Transition Systems



Bisimilar after “determinization”:



Free PCA
{1}

$[0, 1], +_p$

The Algebraic Theory of Convex Algebras

Positive

$$s, t ::= s +_p t, \quad x \in X \quad \text{for all } p \in [0, 1]$$

*

$$\begin{aligned} (x +_q y) +_p z &\stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\ x +_p y &\stackrel{(C_p)}{=} y +_{1-p} x \\ x +_p x &\stackrel{(I_p)}{=} x \end{aligned}$$

projection: $x +_1 y = x$

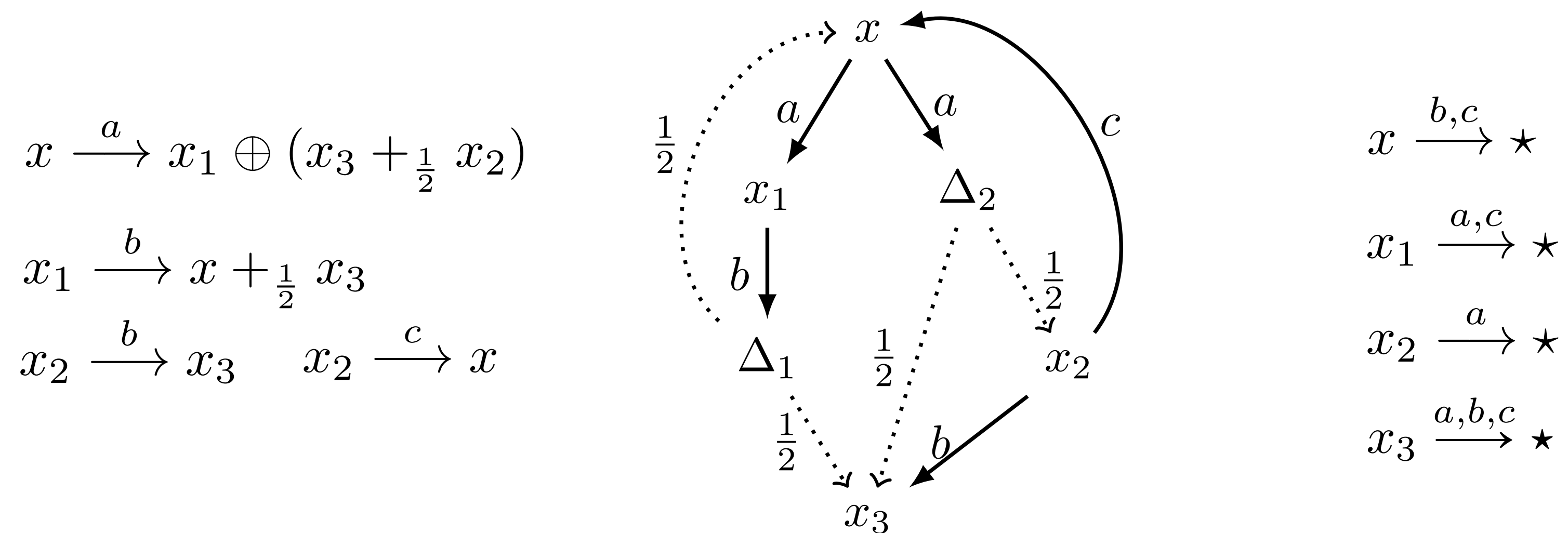
The set of terms quotiented by these axioms is isomorphic to $\mathcal{D}X$

this theory is a presentation for the distribution monad

sub

finitely supported

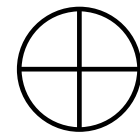
How about probability and nondeterminism (and termination) ?



$$x \xrightarrow{a} x_1 \oplus (x_3 + \frac{1}{2} x_2) \xrightarrow{b} (x + \frac{1}{2} x_3) \oplus (\star + \frac{1}{2} x_3)$$

Three Algebraic Theories

Nondeterminism



$$\begin{array}{lcl}
 (x \oplus y) \oplus z & \stackrel{(A)}{=} & x \oplus (y \oplus z) \\
 x \oplus y & \stackrel{(C)}{=} & y \oplus x \\
 x \oplus x & \stackrel{(I)}{=} & x
 \end{array}$$

Monad: $\mathcal{P}_{ne}$

Algebras: **Semilattices**

Probability $+_p$

$$\begin{array}{lcl}
 (x +_q y) +_p z & \stackrel{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\
 x +_p y & \stackrel{(C_p)}{=} & y +_{1-p} x \\
 x +_p x & \stackrel{(I_p)}{=} & x
 \end{array}$$

Monad: $\mathcal{D}$

Algebras: **Convex Algebras**

Termination ★

no axioms

Monad: $\cdot + 1$

Algebras: **Pointed Sets**

The Algebraic Theory of Convex Semilattices

$$\oplus \quad +_p$$

$$\begin{array}{lll}
 (x \oplus y) \oplus z & \stackrel{(A)}{=} & x \oplus (y \oplus z) \\
 x \oplus y & \stackrel{(C)}{=} & y \oplus x \\
 x \oplus x & \stackrel{(I)}{=} & x \\
 (x +_q y) +_p z & \stackrel{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\
 x +_p y & \stackrel{(C_p)}{=} & y +_{1-p} x \\
 x +_p x & \stackrel{(I_p)}{=} & x \\
 (x \oplus y) +_p z & \stackrel{(D)}{=} & (x +_p z) \oplus (y +_p z)
 \end{array}$$

Monad C : non-empty f.g. convex subsets of distributions

convexity comes from the following derived law

$$s \oplus t \stackrel{(C)}{=} s \oplus t \oplus s +_p t$$

Adding Termination

$$\oplus \quad +_p \quad \star$$

$$\begin{array}{lll}
 (x \oplus y) \oplus z & \stackrel{(A)}{=} & x \oplus (y \oplus z) \\
 x \oplus y & \stackrel{(C)}{=} & y \oplus x \\
 x \oplus x & \stackrel{(I)}{=} & x \\
 (x +_q y) +_p z & \stackrel{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\
 x +_p y & \stackrel{(C_p)}{=} & y +_{1-p} x \\
 x +_p x & \stackrel{(I_p)}{=} & x \\
 (x \oplus y) +_p z & \stackrel{(D)}{=} & (x +_p z) \oplus (y +_p z)
 \end{array}$$

The Algebraic Theory of Pointed Convex Semilattices

$$x \oplus \star \stackrel{(B)}{=} x$$

**The Algebraic Theory of
Convex Semilattices with Bottom**

$$x \oplus \star \stackrel{(T)}{=} \star$$

**The Algebraic Theory of
Convex Semilattices with Top**

Simplified trace

Kurz, Milius,
Pattinson, Schröder

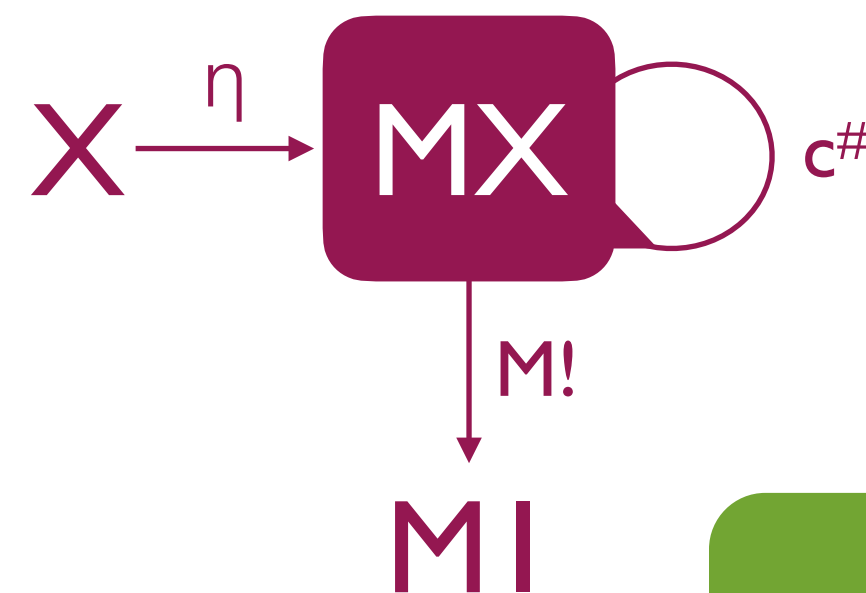
just a monad!

M-coalgebras

instance of a determinisation

$$c: X \rightarrow MX, \quad c^\#: \underline{MX} \xrightarrow{Mc} \underline{MMX} \xrightarrow{\mu} MX$$

Trace can be described using:



$$tr: X \rightarrow (M1)^{\mathbb{N}}$$

works for many generative
examples, as they can be
embedded

$$X \rightarrow \mathcal{P}(A \times X) \rightarrow \mathcal{P}(A^* \times X)$$

monad!

How about reactive / automata ?

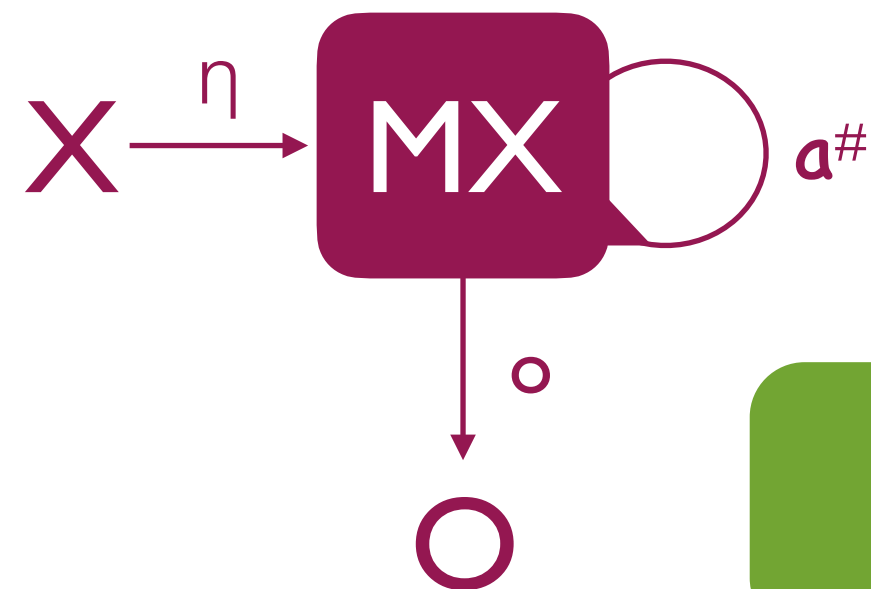
$$\frac{\frac{X \rightarrow O \times (MX)^A}{X \xrightarrow{O} O, \quad X \times A \rightarrow MX}}{X \xrightarrow{O} O, \quad A \rightarrow (X \rightarrow MX)}$$

instance of a determinisation

For each $\mathbf{a} \in A$, an M -coalgebra

$$\mathbf{a}: X \rightarrow MX, \quad \mathbf{a}^\#: \underline{MX} \xrightarrow{M\mathbf{a}} \underline{MMX} \xrightarrow{\mu} MX$$

Trace can be described using:



$$tr: X \rightarrow O^{A^*}$$

$$tr(x)(a_1 \dots a_n) = o \circ a_n^\# \circ \dots \circ a_1^\# \circ \eta$$

extends to words
(A^* is the free monoid)

the syntactic
monoid from
algebraic automata theory

How about reactive / automata ?

$$X \rightarrow O \times (MX)^A$$

$$X \xrightarrow{o} O, \quad X \times A \rightarrow MX$$

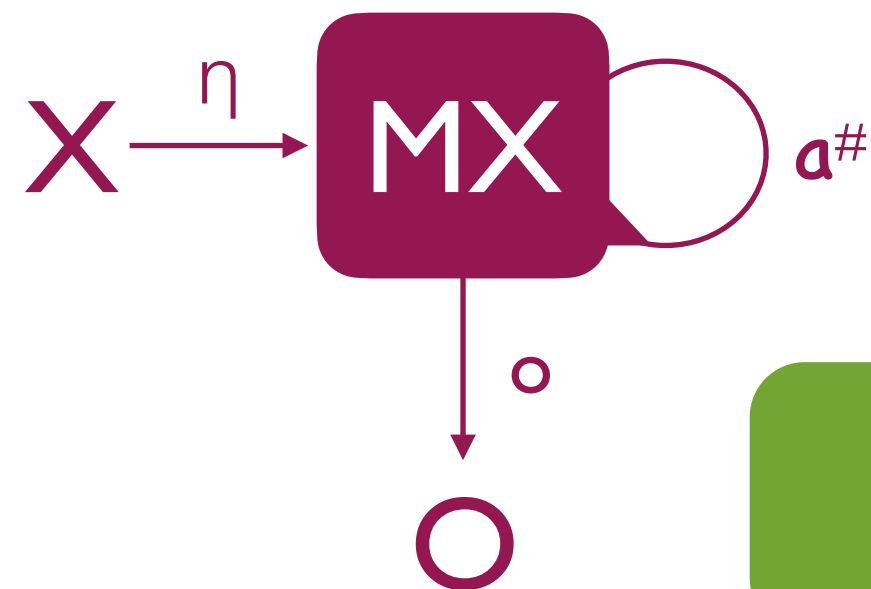
$$X \xrightarrow{o} O, \quad A \rightarrow (X \rightarrow MX)$$

instance of a determinisation

For each $\mathbf{a} \in A$, an M-coalgebra

$$\mathbf{a}: X \rightarrow MX, \quad \mathbf{a}^\#: \underline{MX} \xrightarrow{M\mathbf{a}} \underline{MMX} \xrightarrow{\mu} MX$$

Trace can be described using:



$$tr: X \rightarrow O^{A^*}$$

$$tr(x)(w) = o \circ w^\# \circ \eta$$

extends to words
(A^* is the free monoid)

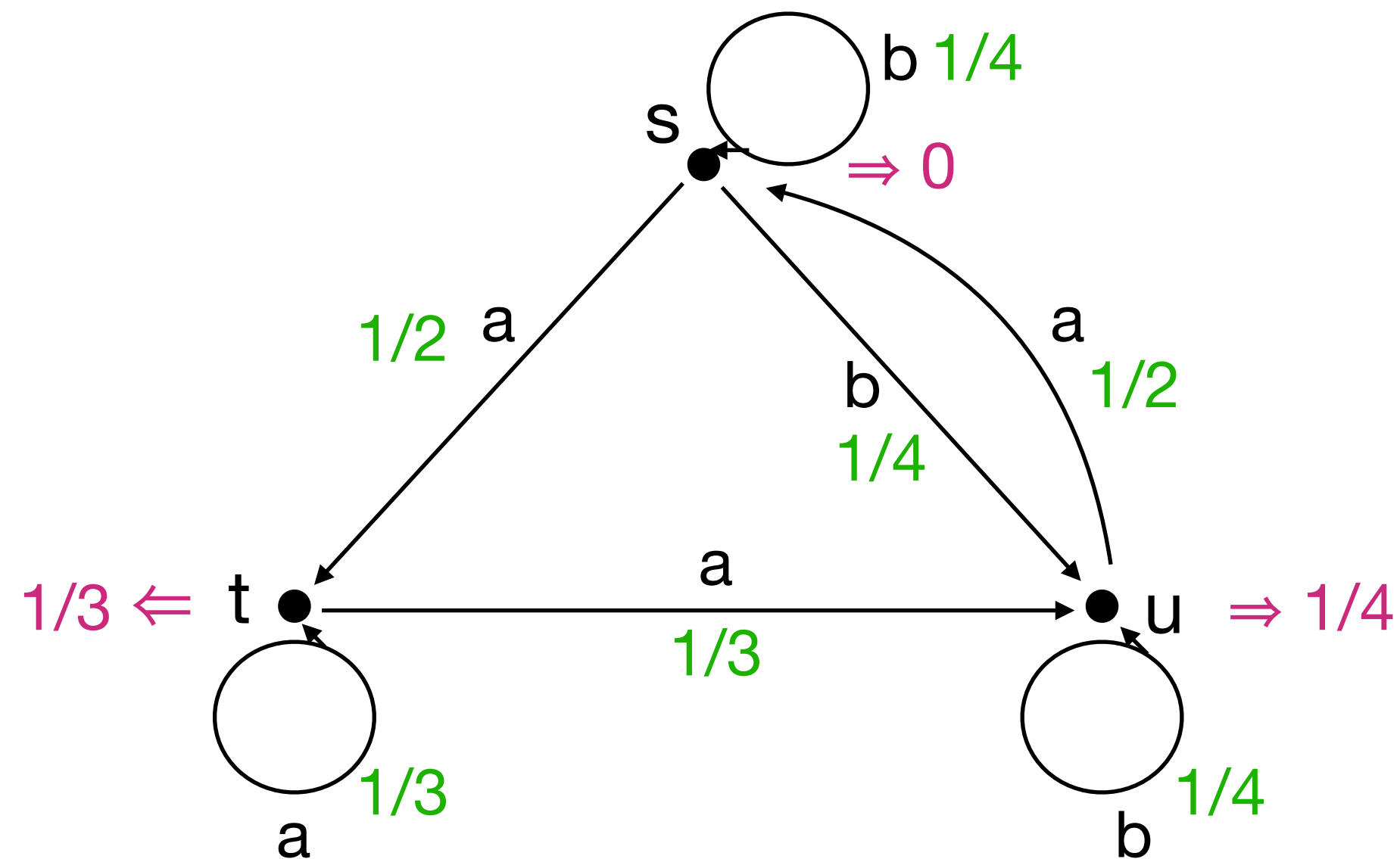
the syntactic monoid
from
algebraic automata theory

Expressions and Axioms



Finite Traces

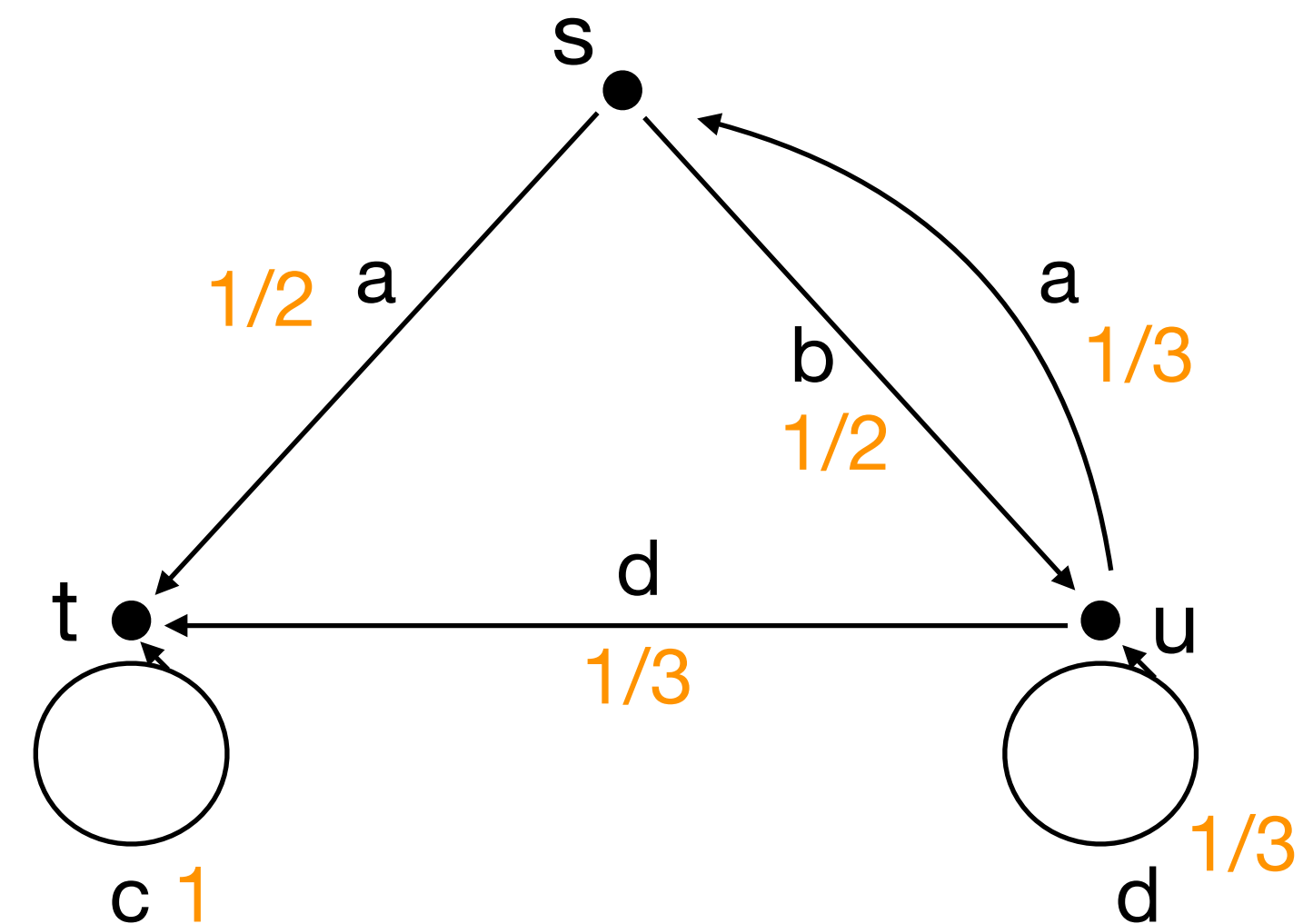
$$c: S \rightarrow \mathcal{D}_{\leq}(A \times S)$$



$$\begin{aligned} \text{trace}(s)(aaa) &= \text{Prob}(\underline{\text{sttt}}, \underline{\text{sttu}}, \underline{\text{stus}}) \\ &= 1/2 \cdot 1/3 \cdot 1/3 \cdot 1/3 \\ &\quad + 1/2 \cdot 1/3 \cdot 1/3 \cdot 1/4 \\ &\quad + 1/2 \cdot 1/3 \cdot 1/2 \cdot 0 \end{aligned}$$

Infinite Traces

$$c: S \rightarrow \mathcal{D}(A \times S)$$



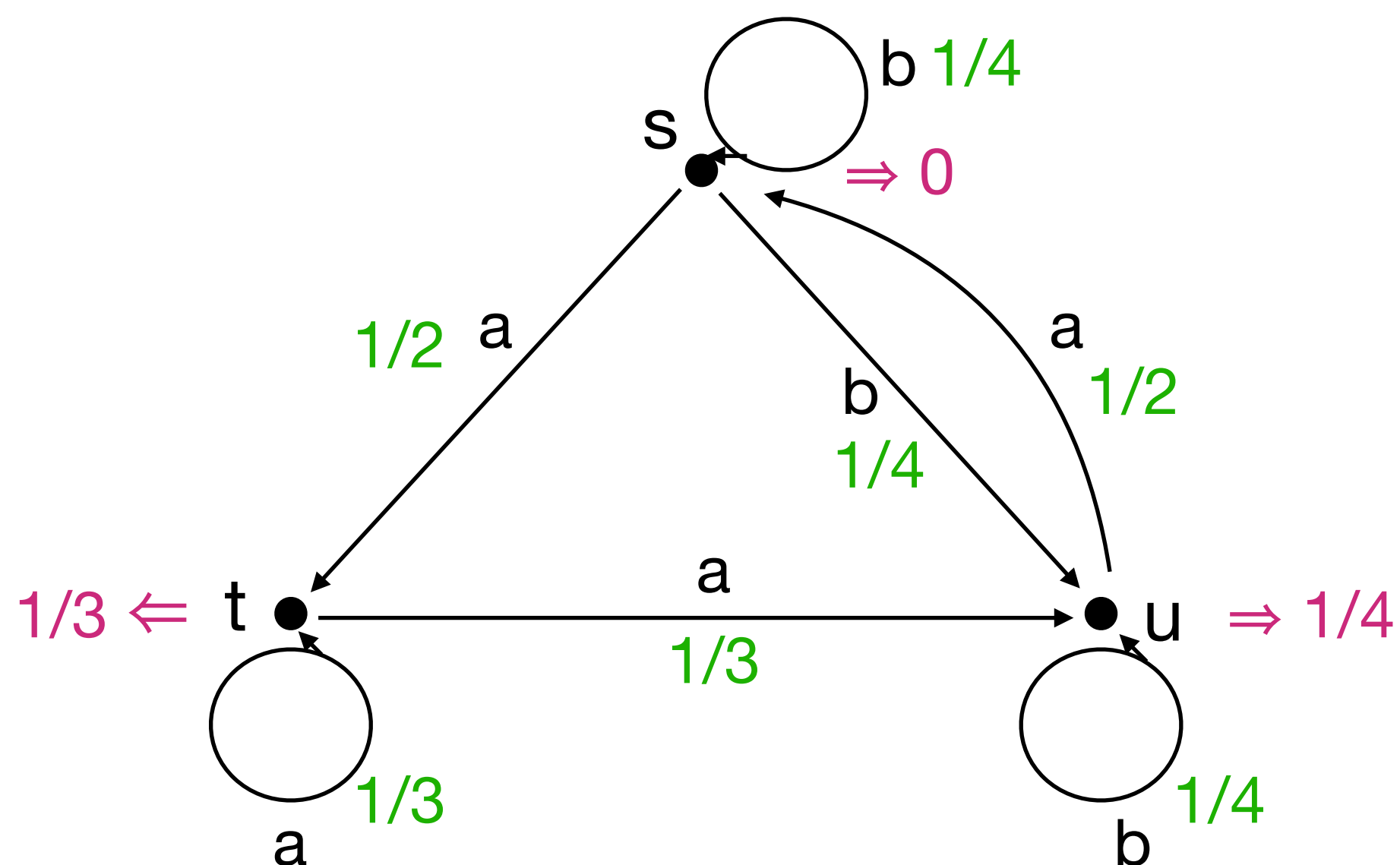
$$\text{trace}(s)(ac^\omega) = 1/2$$

$$\text{trace}(s)(bd^\omega) = \lim_{n \rightarrow \infty} 1/2 \cdot (1/3)^n = 0$$

Therefore, measurable spaces !

Finite Traces

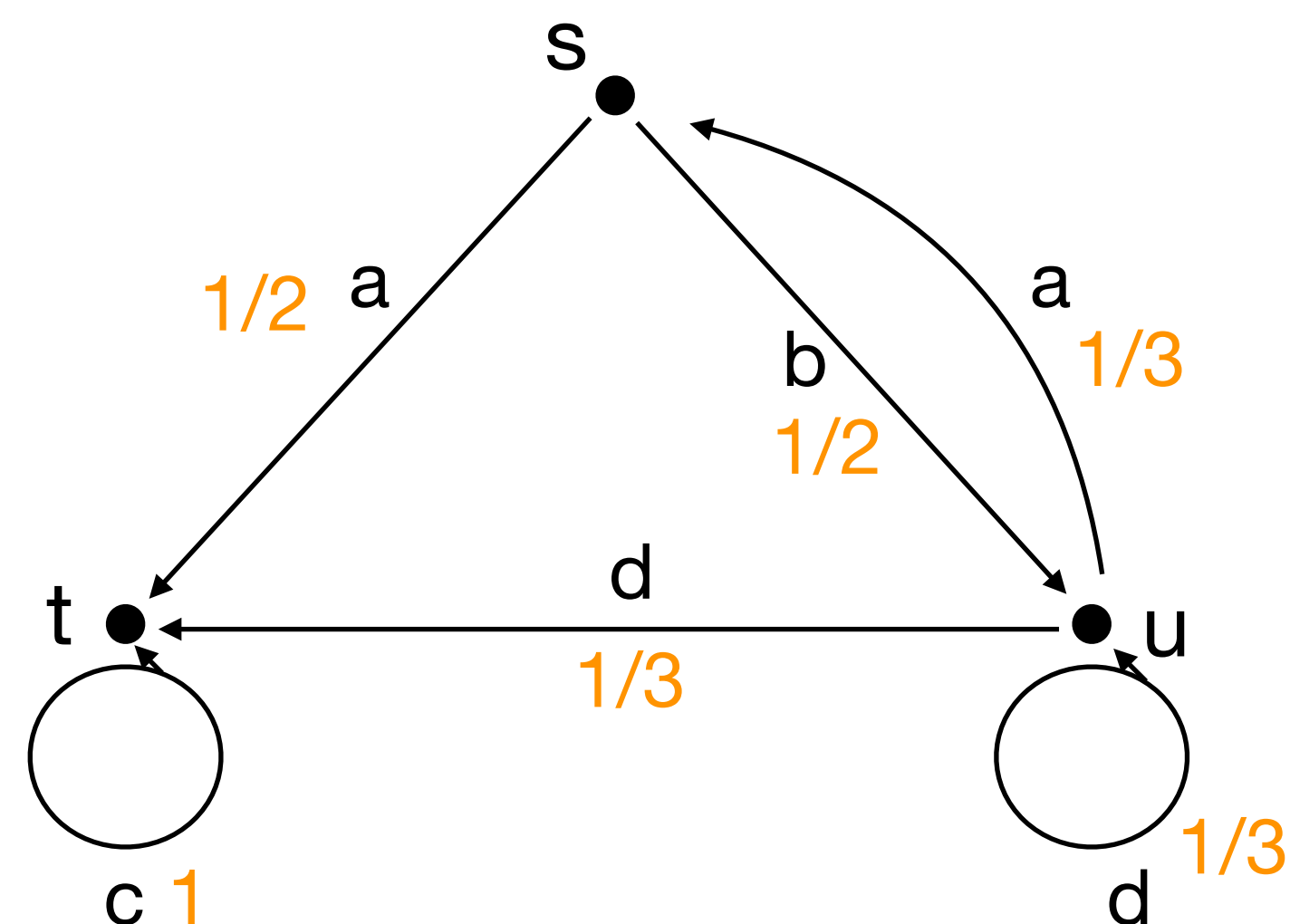
$$c: S \rightarrow \mathcal{D}_{\leq}(A \times S)$$



$$\begin{aligned} \text{trace}(s)(aaa) &= \text{Prob}(\underline{\text{sttt}}, \underline{\text{sttu}}, \underline{\text{stus}}) \\ &= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \\ &\quad + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{4} \\ &\quad + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 0 \end{aligned}$$

Infinite Traces

$$c: S \rightarrow \mathcal{D}(A \times S)$$

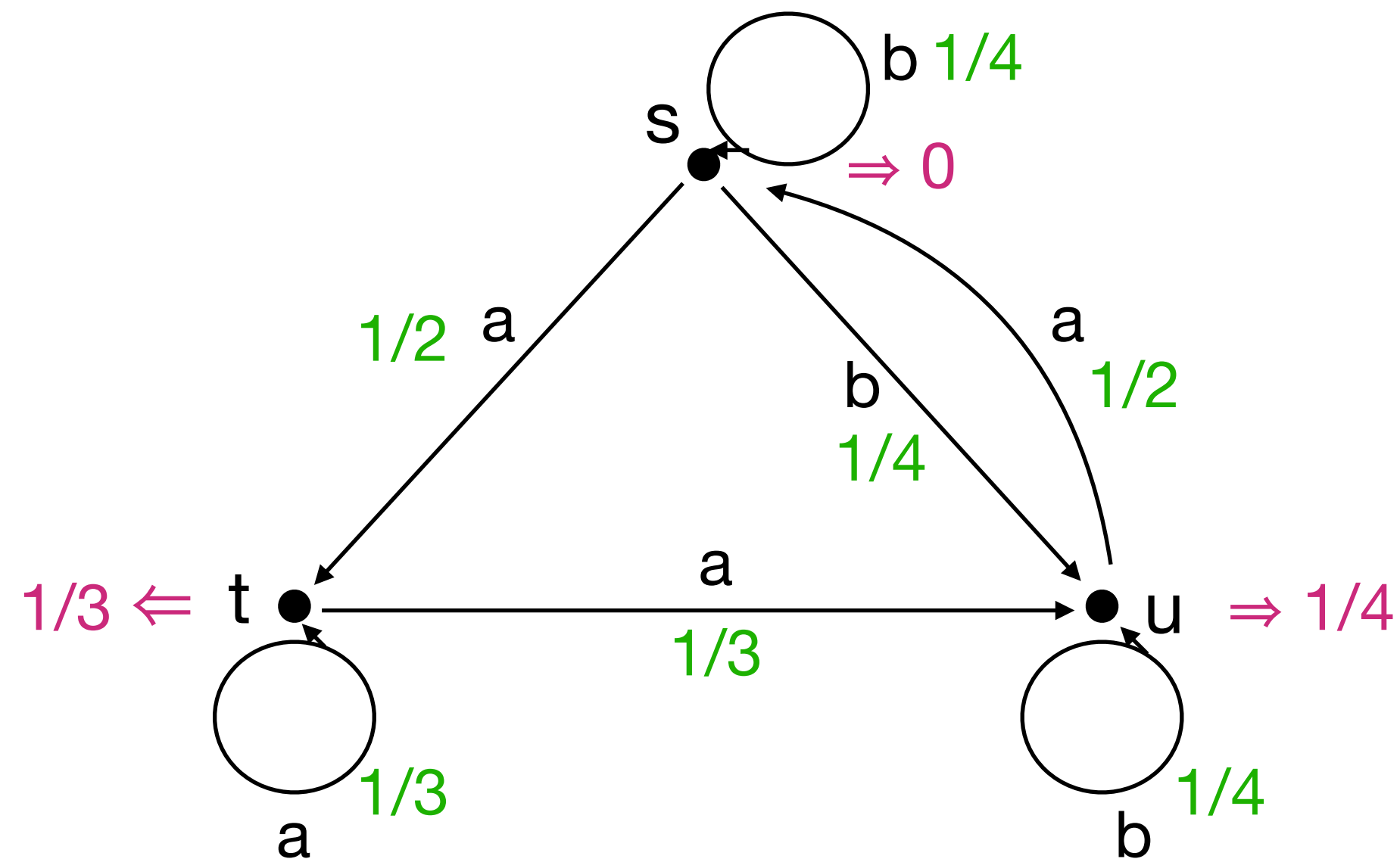


Each state defines a Borel probability measure on A^ω , $\text{trace}(s) \in \text{Prob}(A^\omega)$

with the σ -algebra generated by the cylinders $B_w = \{ \alpha \in A^\omega \mid w \text{ is a prefix of } \alpha \}$

Finite Traces

$$c: S \rightarrow \mathcal{D}_{\leq}(A \times S)$$

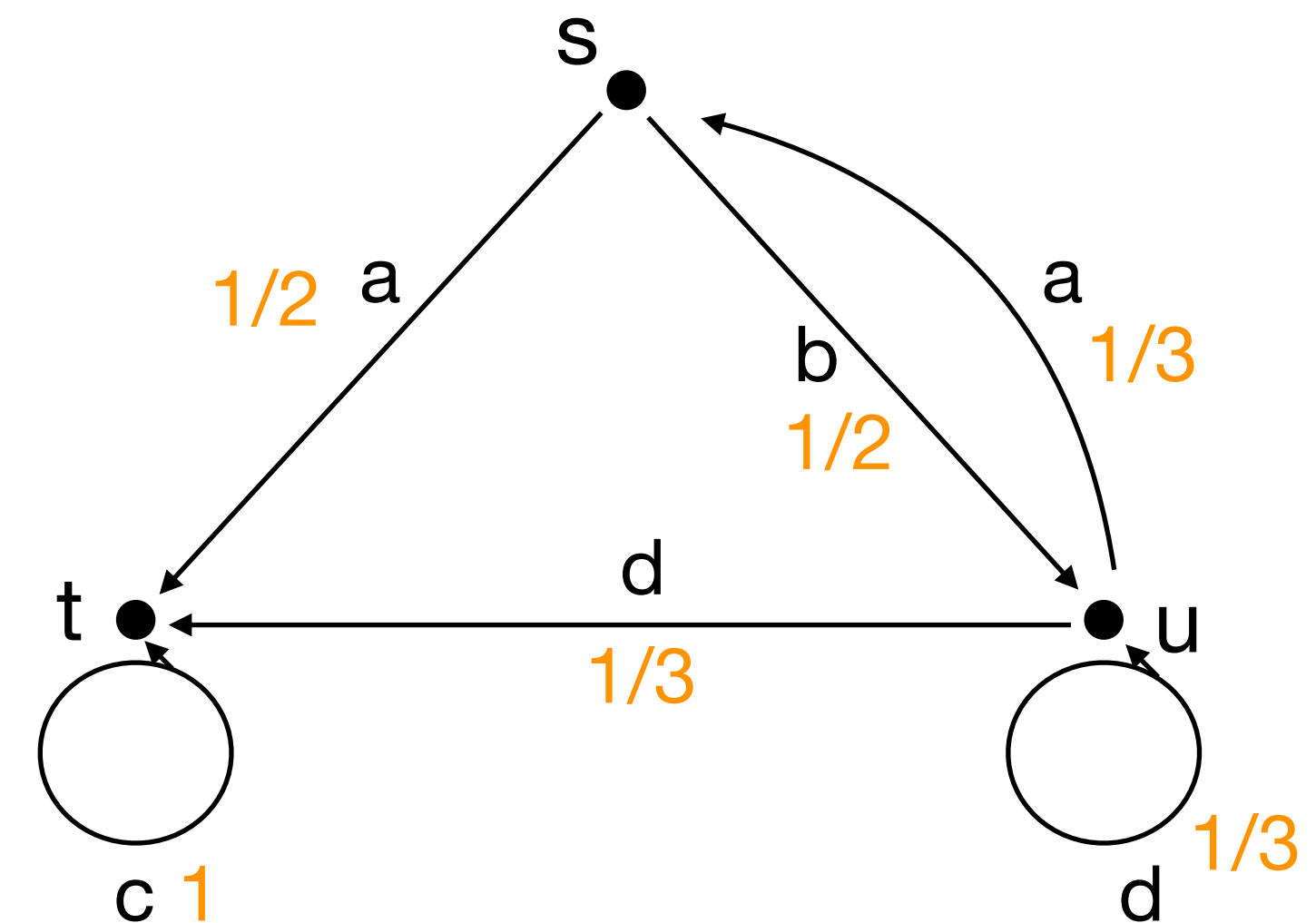


$$\text{trace}(s)(\varepsilon) = c(s)(*) = \text{output-at-}s$$

$$\text{trace}(s)(aw) = \sum_{(a,t) \in \text{supp}(s)} c(s)(a,t) \cdot \text{trace}(t)(w)$$

Infinite Traces

$$c: S \rightarrow \mathcal{D}(A \times S)$$



$$\text{trace}(s)(B_{\varepsilon}) = 1$$

$$\text{trace}(s)(B_{aw}) = \sum_{(a,t) \in \text{supp}(s)} c(s)(a,t) \cdot \text{trace}(t)(B_w)$$

Expressions

Expressions for prob. transition systems: Stark & Smolka '00

$e, f ::= v \mid e \oplus_p f \mid a.e \mid \text{fix } v.e \mid ^*$

for $p \in [0,1]$, $a \in A$

Process terms are expressions that are guarded and closed.

every v in
scope of $a.(-)$

every v in
scope of $\text{fix } v.(-)$

Process terms behave:

$\gamma : \text{PTerms} \rightarrow \mathcal{D}(A \times \text{PTerms})$

Theorem (Kleene-style) : For every state s in a PTS, there exists a process term t_s with

$$s =_{\text{tr}} t_s$$

Axioms

[Cirstea, Moss, Noquez,
Schmid, Silva, S. — CSL'25]

Axioms for (infinite) trace semantics of prob. transition systems:

$$\begin{aligned} e \oplus_1 f &= e \\ e \oplus_p e &= e \\ e \oplus_p f &= f \oplus_{1-p} e \\ e \oplus_p (f \oplus_q g) &= (e \oplus_{pq} f) \oplus_{q(1-p)/(1-pq)} g \end{aligned}$$

convex algebras

Basically the same from
Silva & S. '11

$$a. (e \oplus_p f) = a.e \oplus_p a.f$$

distributivity

$$\begin{aligned} \text{fix } v.e &= e[v := \text{fix } v.e] \\ g = e[v := g] &\Rightarrow g = \text{fix } v.e \end{aligned}$$

fixpoints

Theorem (Soundness & Completeness):

Two states s and t in a PTS are trace equivalent iff
their corresponding process terms are provably equivalent

$$s =_{\text{tr}} t \iff \vdash t_s = t_t$$

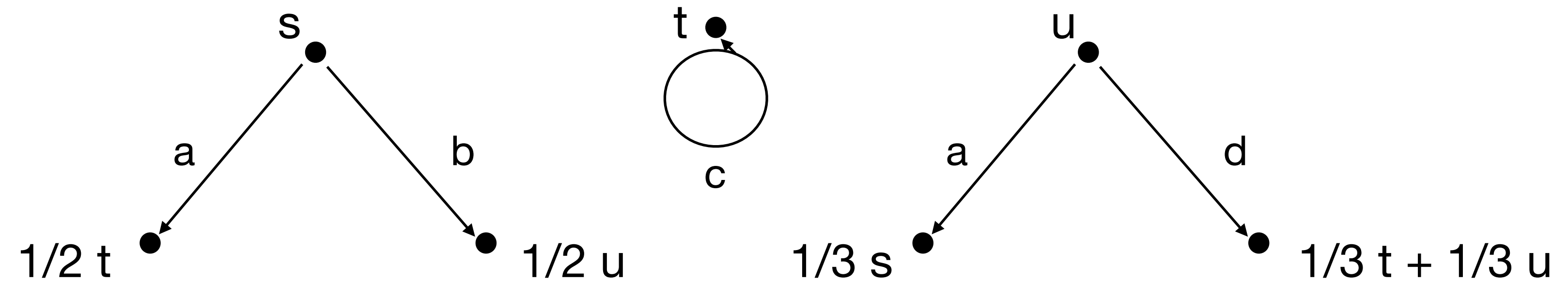
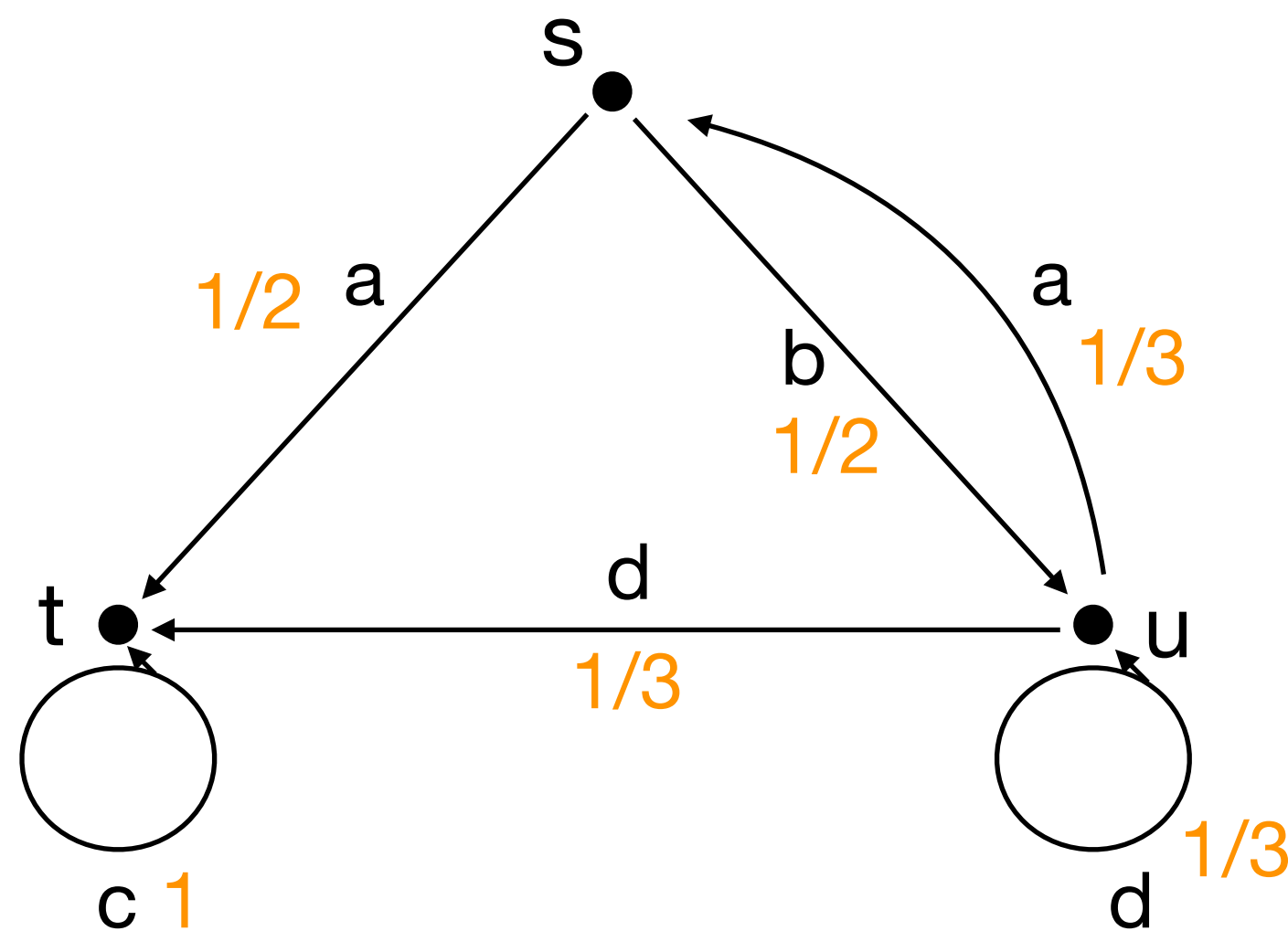
The Role of Algebras

A lot of work happens in categories of algebras, for LMC - convex algebras

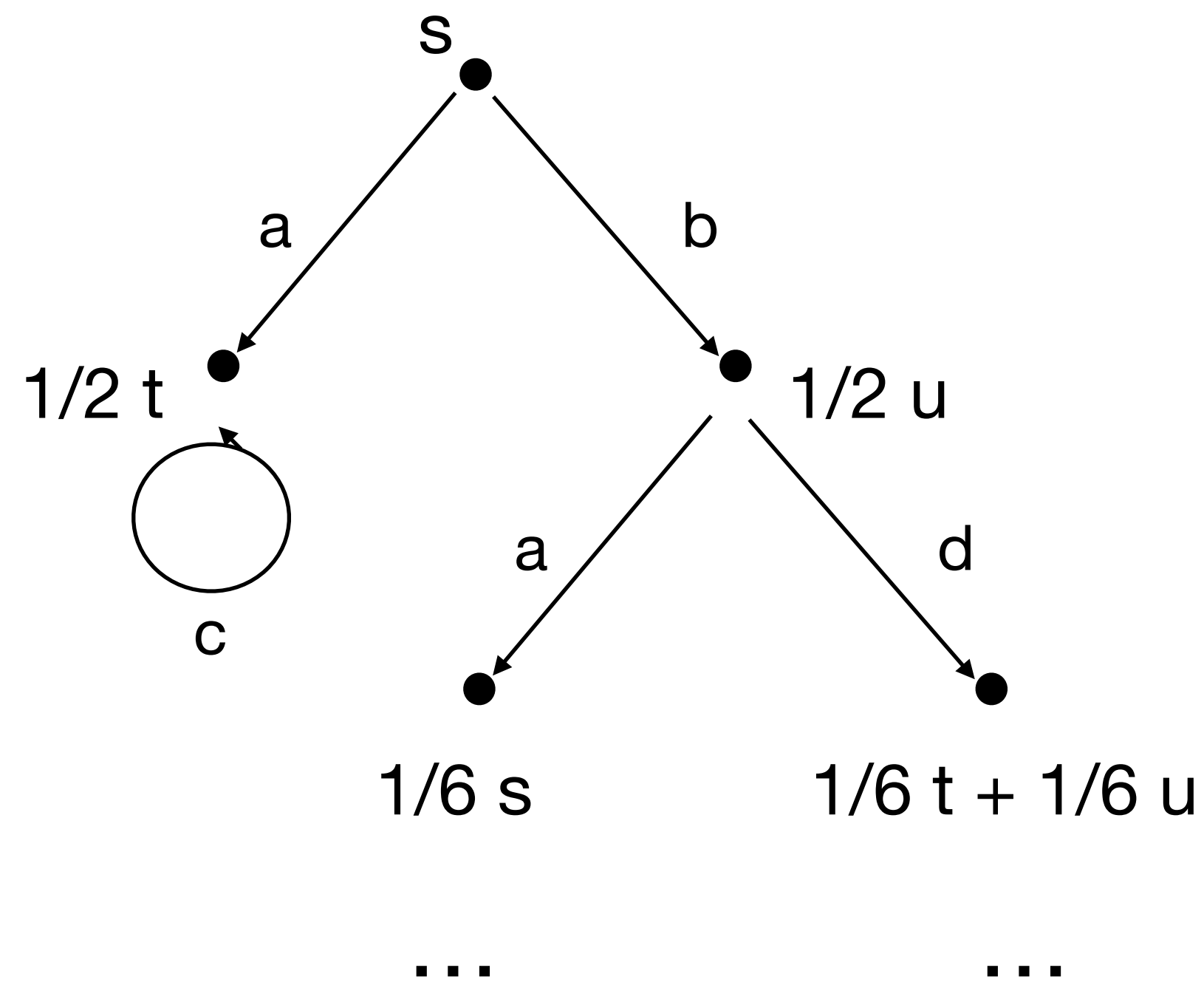
Eilenberg-Moore algebras for $\mathcal{D}$

“Determinization” of

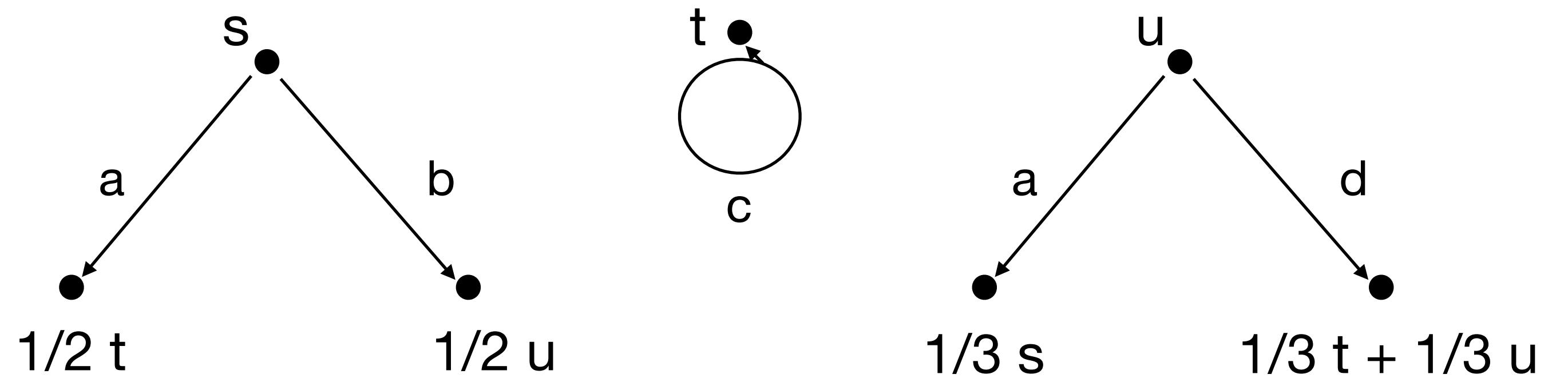
Freely generated by:



Example Determinization

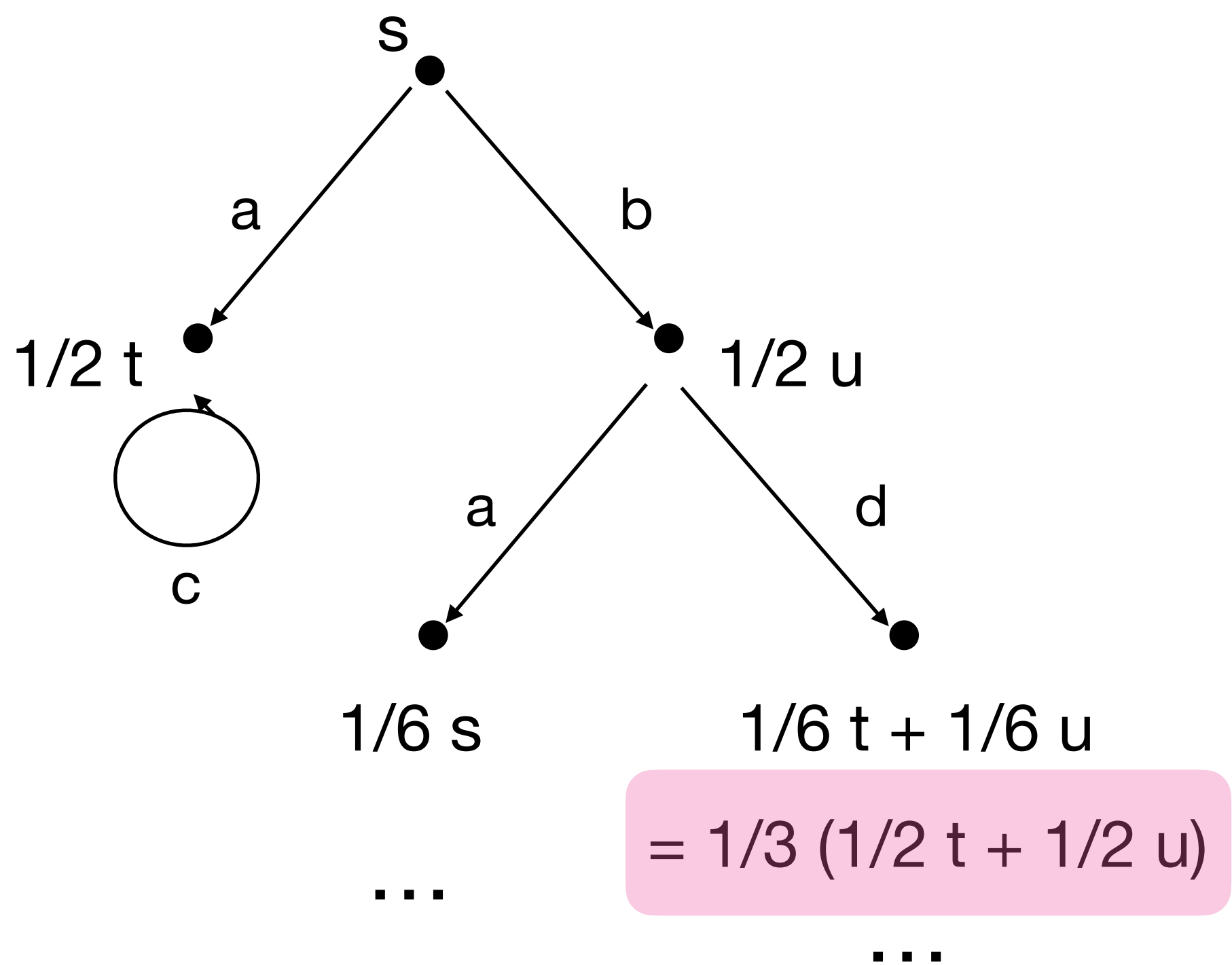


Freely generated by:

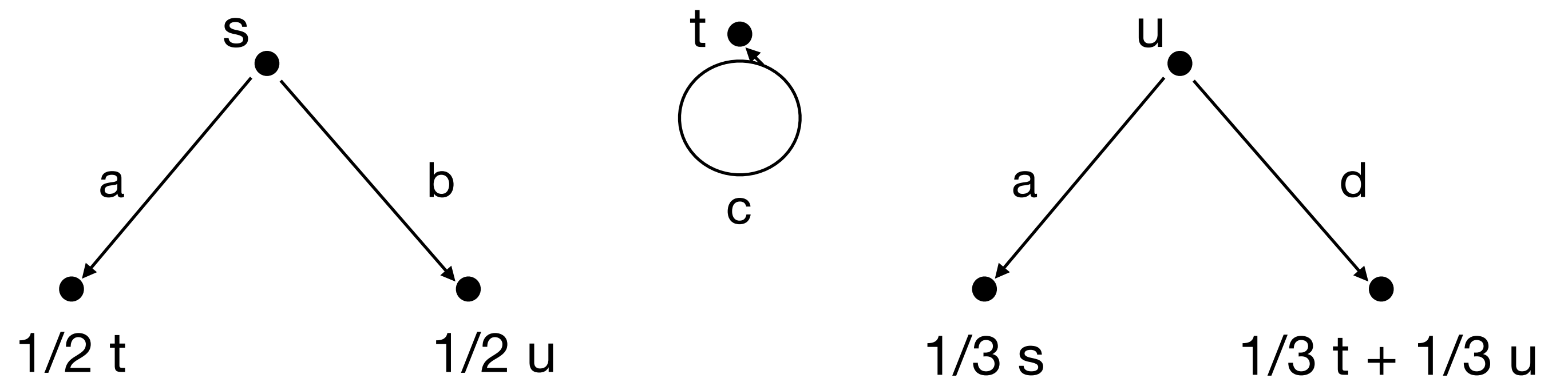


There are different ways to represent this as a coalgebra $\mathcal{D}_{\leq} S \rightarrow G\mathcal{D}_{\leq} S$

The Example Determinization



Freely generated by:



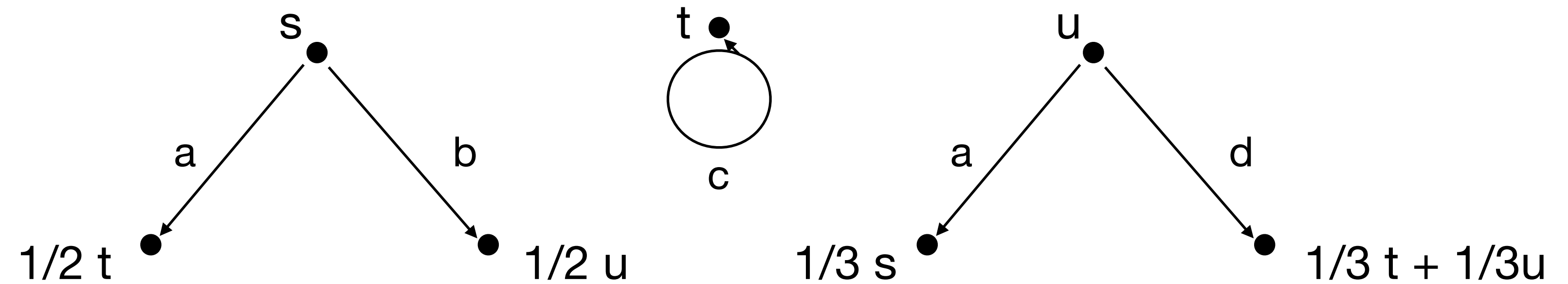
The role of G

1. Every coalgebra $c: S \rightarrow \mathcal{D}(A \times S)$ determinizes to $c^\#: \mathcal{D}S \rightarrow G\mathcal{D}S$

Freely generated by:

2. $\text{Prob}(A^\omega)$ carries the final G-coalgebra

$\text{Prob}(A^\omega) \rightarrow G \text{Prob}(A^\omega)$



3. Infinite trace equivalence is the final coalgebra semantics

There are different ways to represent this as a coalgebra $\mathcal{D}_\leq S \rightarrow G\mathcal{D}_\leq S$

The functor G

First functor on Convex Algebras:

$$X_{\perp} = \{\perp\} \cup \{r \cdot x \mid r \in (0,1], x \in X\}$$

adds a free generator to X

convention: $\perp = 0 \cdot -$

Now G is just an exponent:

$$GX = \{ f : A \rightarrow X_{\perp} \mid f(a) = r_a \cdot x, \sum_{a \in A} r_a = 1 \}$$

Idea (total weight) from
Goy & Rot '18

Some More History

Milius '12 If $fp = fg$ and , then “completeness”

S. & Woracek '15 $fp = fg$ for convex algebras, but no 

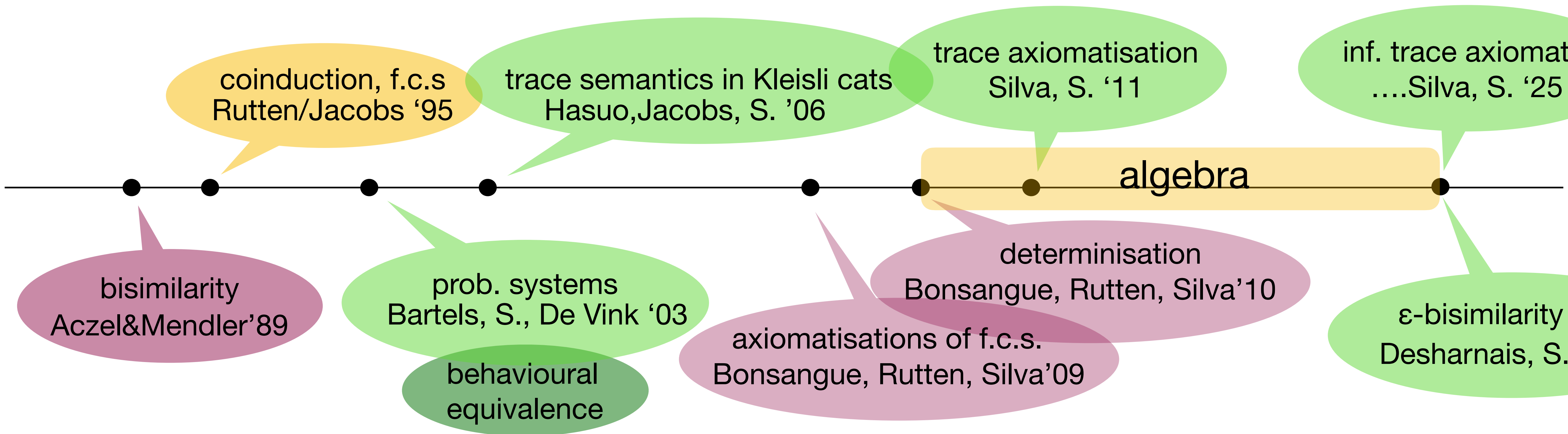
Milius '13/'17 If proper functor and , then “completeness”

S. & Woracek '18 Proper functor(s)  hard proof

Now, for infinite traces: Remarkably beautiful and easy proof of properness

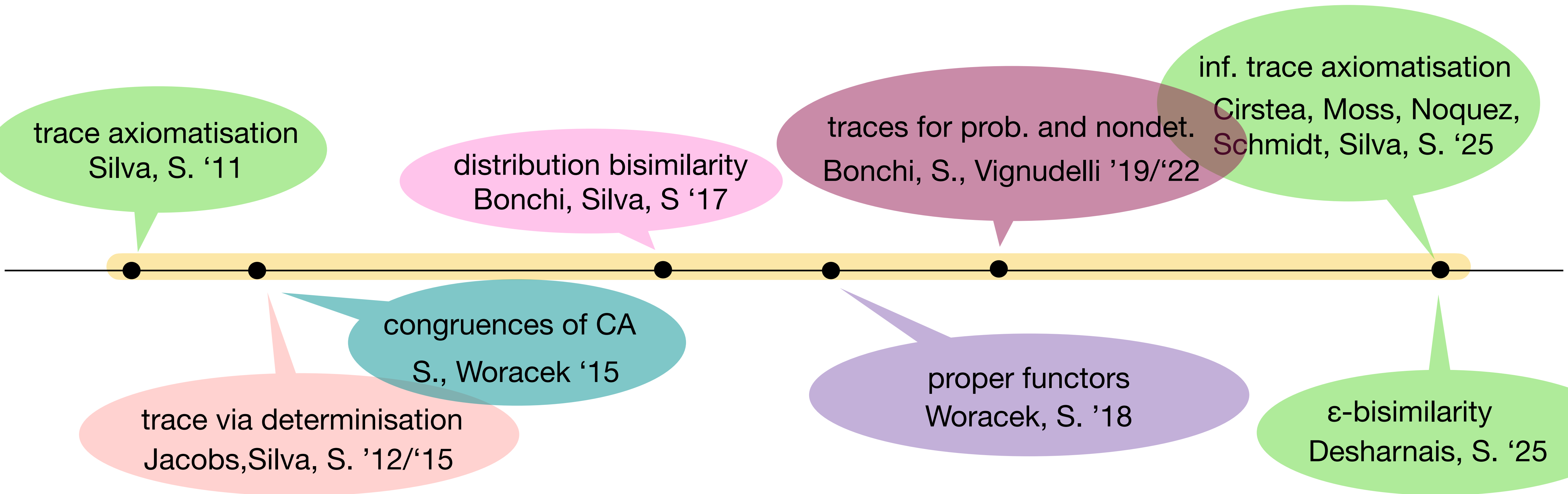
Bisimilarity gives the zigzag

My Simplified History of Coalgebraic Semantics



bisimilarity = behavioural equivalence = final coalgebra semantics (for wpp functors)

My Brief History of Coalgebra over Algebra



III. Logics, Distances, ... via Liftings

Unfortunately, there remained little time — so just a tiny, brief, introduction.

Thank You

