Small Dialectica Categories

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Abstract

More than thirty years after the introduction of Dialectica categories as categorical models of linear logic, it is worth asking a simple question: *why does the Dialectica construction work so well?* This talk argues that many of the answers are already visible in its smallest nontrivial instances. The Dialectica construction, inspired by Gödel's Dialectica interpretation, has become a versatile method for constructing categorical models of substructural logics. It comes in two main variants, **GC** and **DC**, and has been used to provide semantics for intuitionistic and classical linear logic, the Lambek calculus, and numerous related systems. Despite its elegance and wide applicability, the construction is often regarded as technically sophisticated, making it difficult to isolate the essential algebraic mechanisms responsible for its success.

This talk studies what we call *small Dialectica categories*: Dialectica categories constructed over finite partially ordered sets, including the two-element lattice and finite chains. These examples retain the characteristic operations of the full construction while remaining sufficiently concrete that every object, morphism, and categorical operation can be described explicitly. Even extremely small ordered structures generate surprisingly rich categorical behaviour: the specialization of the Dialectica construction to partially ordered sets exposes, in an elementary setting, the interaction between Heyting algebras, residuated lattices, and models of linear logic. Varying the underlying order makes visible how algebraic properties of the base determine the tensor product, linear implication, duality, and other categorical structure of the resulting model.

These finite models provide a laboratory for investigating which logical principles depend on particular algebraic assumptions and which arise uniformly across all instances of the construction. In particular, they offer a new perspective on the *two-dimensional parameter space* of Dialectica constructions: varying the truth-value object L (from Boolean $\mathbf{2}$ to Kleene $\mathbf{3}$, natural numbers $\mathbb{N}$, Lawvere's $[0,\infty]^{\mathrm{op}}$, or a general lineale L) and varying the base category $\mathbf{C}$ (from the walking arrow $\mathbf{2}$ to $\mathbf{Set}$) produces different logical systems with different structural properties. We hope that working out the algebra explicitly in the very small cases will explain structurally *why* this is so, and in particular illuminate where the ! modality comes from and why it behaves differently in **GC** and **DC**.

References

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[*] Joint work with Colin Bloomfield and Peter Jipsen.