

Topology and Metric

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The title of this talk is reminiscent to Nachbin’s fundamental monograph “Topology and Order” [18] where he investigates, among other topics, order-topological versions of normality and uniformity. Since Lawvere’s [16] rather surprising presentation of metric spaces as enriched categories – or, more modestly, as enriched ordered sets – several attempts have been made to investigate metric and other types of spaces as “enriched ordered sets”, i.e. “ordered sets with truth values in $[0, \infty]$ ”. One particular (in our opinion) interesting aspect is developed in [20] by bringing Nachbin’s ordered topological structures [18] into the realm of metric spaces and further to quantale-enriched categories [12]. We recall that an ordered compact Hausdorff space is given by a compact Hausdorff space X also equipped with an order relation which is compatible with the topology; this compatibility can be expressed by saying that the characteristic map $X \times X \rightarrow 2$ of the order relation is continuous where 2 denotes the Sierpiński space. Similarly, a metric compact Hausdorff space is given by a set equipped with a metric and with a compact Hausdorff topology which is not necessarily induced by the metric but should be compatible, meaning that the metric $d: X \times X \rightarrow [0, \infty]$ is lower semicontinuous. With different motivation, similar combinations of topology and metric were also investigated in [17, 4, 5, 19].

It is now tempting to lift several aspects of “topology and order” to the metric or even quantale-enriched context: in particular the equivalence of partially ordered compact spaces and stably compact spaces (see, for instance, [15]), Stone-type dualities, Vietories and Hausdorff functors, and domain-theoretic notions and results.

After recalling some basic properties and results, in this talk we will focus on our more recent work regarding duality theory and the study of more specific properties like exponentiable spaces. For instance, it is known since the late 1960’s that the dual of the category of compact Hausdorff spaces and continuous maps is a variety of algebras (see [8, 14]); surprisingly, a corresponding result for ordered compact Hausdorff spaces was only established very recently (see [9, 3, 1]). In this talk we will show that also the dual of the category of metric compact Hausdorff spaces is a variety of algebras (a first result in this direction can be found in [2]), develop a more concrete duality theory for these spaces (a work initiated in [10, 11]), and give a characterisation of those spaces that admit convenient function spaces. An important technical result is a metric version of the classic Urysohn Lemma for normal topological spaces. Hence, in this talk we will also discuss this and related results in the metric context (here we build on [5, 17]).

If time permits, we will also discuss our recent work on *normed categories* [6, 13] which constitutes a completely different approach to “metric spaces as categories” (suggested very briefly in [16] as well).

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