The logic of arithmetic Markov chains

Serafina Lapenta

Department of Mathematics, University of Salerno.
slapenta@unisa.it

One of the key results in the theory of Riesz spaces is *Yosida duality* [3], which states that the category of metrically-complete, Archimedean and unital Riesz spaces is dual to the category of compact Hausdorff spaces. This theorem was part of a large body of results obtained during the 1930s and 1940s.

We introduce *arithmetic Markov chains* and prove that they are dual to modal ℓ-groups, and hence equivalent, via Mundici's equivalence, to *modal MV-algebras*. This yields a modal Łukasiewicz logic for probabilistic transition systems without real scalar connectives, and a unifying framework for two variations of Yosida duality.

The first variation [1] generalizes Yosida duality to *ℓ-groups* through the notion of *arithmetic spaces*: a topological space X with a continuous function $\zeta\colon X \to \mathbb{N}$, where $\mathbb{N}$ carries the upper topology on the divisibility order. Their arrows are *a-maps*, i.e. continuous maps decreasing denominators. The main result [1, Theorem 7.10] is a duality between metrically-complete, Archimedean, unital ℓ-groups and *normal* arithmetic spaces.

The second, "stochastic" variation [2] defines *Markov chains* as coalgebras for a functor R on compact Hausdorff spaces, together with the variety of *modal Riesz spaces*: a Riesz space with a distinguished positive element u and a positive linear modality $\Diamond$ such that $\Diamond(u) \leq u$. The corresponding duality [2, Theorem 5.1] relates metrically-complete, Archimedean, unital modal Riesz spaces and Markov chains.

By replacing modal Riesz spaces by *modal ℓ-groups* —an ℓ-group with a distinguished positive element and a modality $\Diamond$ as above— we are able to remove the continuum of real scalar connectives. The new dual objects are *arithmetic Markov chains*: coalgebras $\alpha\colon (X,\zeta) \to R_a(X,\zeta)$, where $R_a(X,\zeta)$ is the arithmetic space of positive additive subunital functionals on $C_a(X,\mathbb{R})$ and ζ assigns an arithmetic weight to each state. Our setting jointly generalizes both dualities in [2, 1].

Through the link between ℓ-groups and Łukasiewicz logic, the resulting modal system — Łukasiewicz logic with the operator $\Diamond$, which we call *Markov-Łukasiewicz logic*— is algebraizable, with equivalent algebraic semantics the variety of *modal MV-algebras*. Adding an infinitary deduction rule yields soundness and completeness with respect to arithmetic Markov chains. Completeness also applies to the denominator-zero fragment of arithmetic Markov chains, recovering the Markov processes of [2].

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References

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