Pointfree theories of T_0 spaces

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The classical adjunction between frames and topological spaces,

$$\Omega : \mathbf{Top}^{\mathsf{op}} \rightleftarrows \mathbf{Frm} : \mathsf{pt},$$

restricts to a duality between the category of sober spaces and the category of spatial frames. The problem of refining this adjunction so that it captures all T_0 spaces as fixpoints has been considered in recent literature; three different approaches exist (found in [4], [2], [6]).

We abstract from these, and in [5] introduce the category of *spatializable* **Frm**-concrete categories, abbreviated as *SFC-categories*, where objects are pairs $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, \mathbf{2}_{\mathcal{C}})$ such that $\mathcal{O}$ is a faithful functor, and $\mathbf{2}_{\mathcal{C}} \in \mathcal{C}$ is such that $\mathcal{O}\mathbf{2}_{\mathcal{C}} \cong \mathbf{2}$, and forms a dualizing object, in the sense of, e.g. [3], together with the Sierpiński space $\mathbb{S}$. Directly from $(\mathbf{2}_{\mathcal{C}}, \mathbb{S})$ being a dualizing object, we get the following fact.

Proposition 1. *There is a full subcategory $\mathbf{Top}_{\mathcal{C}} \subseteq \mathbf{Top}$ and a natural contravariant adjunction $\mathsf{pt}_{\mathcal{C}} : \mathcal{C} \leftrightarrows \mathbf{Top}_{\mathcal{C}} : \Omega_{\mathcal{C}}$, as shown below:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathcal{O}"'] & \mathbf{Top}_{\mathcal{C}}^{\mathsf{op}} \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \arrow[d, hook] \\
\mathbf{Frm} \arrow[r, shift left, "\mathsf{pt}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}^{\mathsf{op}} \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

In the right adjoint direction, there is a natural isomorphism $\zeta : \mathcal{O}\Omega_{\mathcal{C}} \cong \Omega$. In the left adjoint direction, there is a componentwise subspace embedding $\xi : \mathsf{pt}_{\mathcal{C}} \to \mathsf{pt}\mathcal{O}$.

We are especially interested in looking at initial and terminal objects of $\mathcal{O}$-fibers, inspired by the following result by Banaschewski and Pultr, from [1].

Theorem 2. *For the functor $\Omega : \mathbf{Top_0}^{\mathsf{op}} \to \mathbf{Frm}$:*

- *Sober spaces coincide with the fiber-initial objects;*
- *T_D-spaces coincide with the fiber-terminal objects.*

Taking into account contravariance of Ω, the theorem essentially states that a T_0 space X is sober iff it is the largest among the spaces with an isomorphic frame of opens; it is T_D iff it is the smallest. We will recover a general version of this theorem in our setting. We say that an SFC-category is *total* if

1. All T_0 spaces are fixpoints of $\mathsf{pt}_{\mathcal{C}} \dashv \Omega_{\mathcal{C}}$;
2. The fibers of $\mathbf{2}$ and $\mathbf{3} = \Omega(\mathbb{S})$ only contain one object, up to isomorphism;

3. The functor $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ has a left adjoint $\mathcal{F} : \mathbf{Frm} \to \mathcal{C}$ with $\mathcal{O}\mathcal{F}$ an isomorphism;
4. The spatialization $\Omega_{\mathcal{C}}\mathsf{pt}_{\mathcal{C}}C$ of a fiber-terminal object C is fiber-terminal;
5. All locally closed sublocales of a frame $\mathcal{O}C$ lift to $\mathcal{C}$.

We obtain the following generalization of the result by Banaschewski and Pultr.

Theorem 3. *Let $(\mathcal{O} : \mathcal{C} \to \mathbf{Frm}, \mathbf{2}_{\mathcal{C}})$ be a total SFC-category. Let $\mathcal{C}_{\mathcal{I}} \subseteq \mathcal{C}$ and $\mathcal{C}_{\mathcal{T}} \subseteq \mathcal{C}$ be the subcategories of fiber-initials and fiber-terminals, respectively. The dualizing object $(\mathbf{2}_{\mathcal{C}}, \mathbb{S})$ induces dual adjunctions*

$$\mathsf{pt}_{\mathcal{C}} : \mathcal{C}_{\mathcal{I}} \leftrightarrows \mathbf{Top} : \mathcal{F}\mathcal{O}\Omega_{\mathcal{C}} \qquad\qquad \mathsf{pt}_{\mathcal{C}} : \mathcal{C}_{\mathcal{T}} \leftrightarrows \mathbf{Top_D} : \Omega_{\mathcal{C}}.$$

1. *The fixpoints of the adjunction for fiber-initials coincide with the sober spaces.*
2. *The fixpoints of the adjunction for fiber-terminals coincide with the T_D-spaces.*

We find that there is a chain of morphisms between adjunctions:

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{T}} \arrow[r, shift left, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{Top_D} \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \arrow[d, hook] & \mathsf{Fix}(\mathbf{Top_D}) = \mathbf{Top_D} \\
\mathcal{C} \arrow[r, shift left, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "\mathcal{O}\mathcal{F}"'] & \mathbf{Top} \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \arrow[d, equal] & \mathsf{Fix}(\mathbf{Top}) = \mathbf{Top_0} \\
\mathcal{C}_{\mathcal{I}} \arrow[r, shift left, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left, "\mathcal{F}\mathcal{O}\Omega_{\mathcal{C}}"] & \mathsf{Fix}(\mathbf{Top}) = \mathbf{Sob}
\end{tikzcd}$$

Thus, from any duality for T_0 spaces that satisfies the five natural properties above, one obtains a duality for sober spaces as a quotient, and one for T_D spaces as a subobject.

References

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