Decidability in integral conuclear residuated lattices

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Residuated lattices = substructural generalizations of Boolean algebras.
Conuclei = substructural generalizations of S4 box operators.
Conuclear RLs = substructural generalizations of S4 modal algebras.

Here we consider **only** the box operator and **only** algebraic semantics.

Main decidability result

MV-algebras with a ($\wedge$-)conucleus and Abelian ℓ-groups with a negative ($\wedge$-)conucleus ($\Box x \leq 1$) have a decidable quasi-equational theory.

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MV-algebras with a ($\wedge$-)conucleus and Abelian ℓ-groups with a negative ($\wedge$-)conucleus ($\Box x \leq 1$) have a decidable universal theory.

This contrasts with the **un**decidability result for the basic Łukasiewicz modal logic of all Łukasiewicz **Kripke frames** due to Vidal (2022).

Talk based on the manuscript:

Decidability in classes of integral residuated lattices with a conucleus

I will talk about residuated lattices, but all of the new results – except for those specific to ℓ-groups – live at the level of **ordered universal algebra**. For more details see these slides:

Finite embeddability via nuclear and conuclear images

Both can be found here:

`https://sites.google.com/site/adamprenosil`

Conuclei and conuclear images

A **(bounded) residuated lattice** is a residuated lattice-ordered monoid, i.e. an algebra with:

- lattice operations $x \wedge y$ and $x \vee y$
- (lattice top and bottom $\top$ and $\bot$)
- monoidal multiplication $x \cdot y$ with unit 1
- two division operations $x \backslash y$ and x / y

such that the **residuation law** holds:

$$y \leq x \backslash z \iff x \cdot y \leq z \iff x \leq z / y.$$

Integral: $x \cdot y \leq x$ and $x \cdot y \leq y$ **Commutative**: $x \cdot y = y \cdot x$

EXAMPLE. Lattice-ordered groups (ℓ-groups):

$$y \leq x^{-1} z \iff x \cdot y \leq z \iff x \leq z y^{-1}.$$

EXAMPLE. Boolean and Heyting algebras:

$$y \leq x \rightarrow z \iff x \wedge y \leq z \iff x \leq z \leftarrow y.$$

A **conucleus** on a RL **A** is an order-preserving operation $\Box$ with

$$\Box\Box x = \Box x \leq x, \qquad \Box x \cdot \Box y \leq \Box(x \cdot y), \qquad 1 \leq \Box 1.$$

Equivalently, an interior operator whose image is a submonoid.

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An algebra of the form $\langle \mathbf{A}, \Box \rangle$ will be called **conuclear residuated lattice**. The class of **conuclear expansions** of algebras in a class of RLs K is:

$$\mathbb{C}\mathrm{x}(\mathsf{K}) := \{\langle \mathbf{A}, \Box \rangle \mid \mathbf{A} \in \mathsf{K} \text{ and } \Box \text{ is a conucleus on } \mathbf{A}\}.$$

EXAMPLE. $\mathbb{C}\mathrm{x}$(Boolean algebras) = S4 modal algebras.

The image of a conucleus $\Box$ is a residuated lattice $\mathbf{A}_\Box$ in its own right:

$$\mathbf{A}_\Box := \{a \in \mathbf{A} \mid \Box a = a\} = \Box[\mathbf{A}].$$

The operations $\vee$, $\cdot$, 1 of $\mathbf{A}_\Box$ agree with $\mathbf{A}$. Others operations need not:

$$x \backslash^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x \backslash^{\mathbf{A}} y), \qquad x /^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x /^{\mathbf{A}} y), \qquad x \wedge^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x \wedge^{\mathbf{A}} y).$$

The residuated lattice $\mathbf{A}_\Box$ will be called a **conuclear image** of $\mathbf{A}$.

$$\mathbb{C}(\mathsf{K}) := \{\mathbf{A}_\Box \mid \mathbf{A} \in \mathsf{K} \text{ and } \Box \text{ is a conucleus on } \mathbf{A}\}.$$

Example. $\mathbb{C}$(Boolean algebras) = Heyting algebras.

This is an algebraic formulation of the Gödel translation S4 – IL.

Which universal sentences are preserved by conuclear images?

FACT. All conuclear images preserve universal sentences in $\{\vee, \cdot, 1\}$.

A **$\wedge$-conucleus** is a conucleus which satisfies $\Box(x \wedge y) = \Box x \wedge \Box y$, or equivalently whose image is closed under $\wedge$ (so $x \wedge^{\mathbf{A}_\Box} y = x \wedge^{\mathbf{A}} y$).

FACT. All $\wedge$-conuclear images preserve universal sentences in $\{\wedge, \vee, \cdot, 1\}$.

Montagna and Tsinakis (2010)

- $\mathbb{C}$(Abelian ℓ-groups) $=$ cancellative CRLs.
- $\mathbb{C}_\wedge$(Abelian ℓ-groups) $=$ fully distributive cancellative CRLs.
- $\mathbb{C}$(totally ordered Abelian groups) $=$ totally ordered cancellative CRLs.

A residuated lattice is **cancellative** if it satisfies the quasi-equations

$$z \cdot x = z \cdot y \implies x = y, \qquad x \cdot z = y \cdot z \implies x = y,$$

or equivalently the equations $\quad zx\backslash zy = x\backslash y, \qquad xz/yz = x/y.$

A residuated lattice is **fully distributive** if it satisfies the equations

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z) \qquad (x \wedge y) \cdot z = (x \cdot z) \wedge (y \cdot z)$$

Conuclear RLs and the FEP

PROBLEM. Consider a universal class K of integral residuated lattices.

Does the universal class $\mathbb{C}\mathrm{x}(\mathsf{K})$ have a decidable universal theory?

TRIVIAL. If $\mathbb{C}\mathrm{x}(\mathsf{K})$ has a decidable universal theory, then so does $\mathbb{C}(\mathsf{K})$.

EXAMPLE. $\mathbb{C}(\mathsf{K}) \models (x\backslash y) \vee (y\backslash x) = 1 \iff \mathbb{C}\mathrm{x}(\mathsf{K}) \models \Box(x\backslash y) \vee \Box(y\backslash x) = 1.$

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The default algebraic technique is the Finite Embeddability Property.

A **universal class** of algebras is axiomatized by universal sentences, e.g.:

$$\forall xy\,(x \leq y \text{ or } y \leq x).$$

Theorem. The universal class **generated** by K is $\mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.

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Universal classes and partial embeddings (Maltsev 1973)

Let $\mathbf{A}$ be an algebra and K a class of algebras. Then $\mathbf{A} \in \mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$ if and only if for each finite set $X \subseteq \mathbf{A}$ the finite partial algebra $\mathbf{A}|_X$ has an embedding ι into some $\mathbf{B} \in \mathsf{K}$. (We say that **A locally embeds** into K.)

More explicitly, for each finite $X \subseteq \mathbf{A}$ there is an injective map $\iota : X \hookrightarrow \mathbf{B}$ such that for each n-ary primitive operation $\circ$ and all $a_1, \dots, a_n, b \in X$

$$\circ^{\mathbf{A}}(a_1, \dots, a_n) = b \implies \circ^{\mathbf{B}}(\iota(a_1), \dots, \iota(a_n)) = \iota(b).$$

A class K has the **Finite Embeddability Property** (**FEP**) if each K-algebra locally embeds into a finite K-algebra, or equivalently if $\mathsf{K} \subseteq \mathbb{U}(\mathsf{K}_{\text{fin}})$.

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FEP and decidability (Evans 1969)

If a universal class is finitely axiomatizable and enjoys the FEP (i.e. is generated by finite algebras), then it has a decidable universal theory.

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FEP FOR SUBVARIETIES OF IRLS (Galatos and Jipsen 2013)

So does each variety of integral RLs axiomatized by equations in $\{\vee, \cdot, 1\}$.

QUESTION. If a universal class $\mathsf{K} \subseteq \mathsf{IRL}$ has the FEP, does $\mathbb{C}\mathrm{x}(\mathsf{K})$ inherit it?

EXAMPLE. We know that BA has the FEP. Can we argue as follows?

$$\mathsf{BA} = \mathbb{U}(\mathsf{BA}_{\text{fin}}) \implies \mathbb{C}\mathrm{x}(\mathsf{BA}) = \mathbb{C}\mathrm{x}\mathbb{U}(\mathsf{BA}_{\text{fin}}) = \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{BA}_{\text{fin}})$$

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Key idea: describe $\mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K})$ in terms of local embeddings à la Maltsev.

Consider IRLs $\mathbf{A}$ and $\mathbf{B}$ and a subset $X \subseteq \mathbf{A}$. An embedding $\iota \colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ is a **π-embedding** of the partial algebra $\mathbf{A}|_X$ into $\mathbf{B}$ if for all $x_1, \ldots, x_n, y \in X$

$$x_1 \cdot^{\mathbf{A}} \ldots \cdot^{\mathbf{A}} x_n \leq^{\mathbf{A}} y \iff \iota(x_1) \cdot^{\mathbf{B}} \ldots \cdot^{\mathbf{B}} \iota(x_n) \leq^{\mathbf{B}} \iota(y).$$

That is, the **relative positions** of products w.r.t. $X \subseteq \mathbf{A}$ are preserved.

Conuclear universal classes and π-embeddings

Consider $\mathbf{A} \in \mathsf{IRL}$ and $\mathsf{K} \subseteq \mathsf{IRL}$. Then $\mathbb{C}\mathrm{x}(\mathbf{A}) \subseteq \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K})$ if for each finite set $X \subseteq \mathbf{A}$ the partial algebra $\mathbf{A}|_X$ has a π-embedding into some $\mathbf{B} \in \mathsf{K}$. (We say that **$\mathbf{A}$ locally π-embeds** into K.)

(We could also formulate a necessary and sufficient condition. Less useful.)

Why does this description of $\mathbb{UC}$x hold?

Given an integral RL $\mathbf{A}$ and a finite set $S \subseteq \mathbf{A}$, let

$$\begin{aligned}\langle S\rangle^{\mathbf{A}} &:= \text{the } \{\vee, \cdot, 1\}\text{-subalgebra of } \mathbf{A} \text{ generated by } S \\ &= \text{finite joins of products of elements from } S.\end{aligned}$$

FACT. $\langle S\rangle^{\mathbf{A}}$ is the image of a conucleus $\Box_S$ on (a subalgebra of) $\mathbf{A}$.

FACT. Each conucleus $\Box$ on $\mathbf{A}$ locally looks like some $\Box_S$.

FACT. Each π-embedding ι preserves $\Box_S$ in that $\iota(\Box_S x) = \Box_{\iota[S]}\iota(x)$.

Conuclear K-algebras inherit the FEP

Let $\mathsf{K} \subseteq \mathsf{IRL}$ be a universal class whose monoid reducts are locally finite. (For example, $\mathsf{K} \subseteq \mathsf{ICRL}$ and $\mathsf{K} \vDash x^n = x^{n+1}$.) Then:

$$\mathsf{K} \text{ has the FEP} \iff \mathbb{C}\mathrm{x}(\mathsf{K}) \text{ has the FEP.}$$

Proof. If each $\mathbf{A} \in \mathsf{K}$ locally embeds into K_{fin} and its monoid reduct is locally finite, then $\mathbf{A}$ locally π-embeds into K_{fin}, so $\mathbf{A} \in \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K}_{\text{fin}})$.

Trivial observation. $\mathbb{C}\mathrm{x}(\mathsf{K})$ has the FEP $\implies$ $\mathbb{C}(\mathsf{K})$ has the FEP.

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Beyond the locally finite case, one needs a different argument.

FEP FOR CONUCLEAR ICRLS (Amano 2006)

Conuclear bounded ICRLs (aka $\mathbf{FL_{ew}}$-algebras) have the FEP.

PROOF applies the methodology of Blok & van Alten to conuclear IRLs.

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FEP for universal classes of conuclear IRLs

Each universal class of conuclear integral RLs axiomatized by some set of universal sentences in $\{\vee, \cdot, 1\}$ has the FEP. Same for integral ℓ-monoids.

Proof involves nuclear images and applies equally well to ℓ-monoids. In contrast, G&J rely on residuated frames and assume residuals.

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Syntactic description of conuclear universal classes

A class of IRLs is axiomatized by universal sentences in $\{\vee, \cdot, 1\}$ if and only if it is a universal class ($\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}_{\mathrm{U}}$ closed) closed under conuclear images ($\mathbb{C}$).

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Syntactic description of conuclear quasivarieties

A class of IRLs is axiomatized by quasiequations in $\{\vee, \cdot, 1\}$ if and only if it is a quasivariety ($\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_U$ closed) closed under conuclear images ($\mathbb{C}$).

All new theorems stated so far are in fact **universal algebraic**. For example, swapping monoids for $\wedge$-monoids immediately yields the following results.

FEP for universal classes of $\wedge$-conuclear IRLs

Each universal class of $\wedge$-conuclear **distributive** IRLs axiomatized in $\{\wedge, \vee, \cdot, 1\}$ has the FEP. Same for integral **distributive** ℓ-monoids.

Syntactic description of $\wedge$-conuclear universal classes

A universal class of **distributive** IRLs is axiomatized in $\{\wedge, \vee, \cdot, 1\}$ if and only if it is closed under $\wedge$-conuclear images ($\mathbb{C}_\wedge$).

$\wedge$-conuclear K-algebras inherit the FEP

Let K be a universal class of **distributive** IRLs whose $\wedge$-monoid reducts are locally finite. Then: K has the FEP $\iff$ $\mathbb{C}\mathrm{x}_\wedge(\mathsf{K})$ has the FEP.

Conuclear Abelian ℓ-groups

The problem of finding a generating class for $\mathbb{C}\mathrm{x}(\mathsf{V})$ reduces to looking at finitely subdirectly irreducible algebras (= chains, for Abelian ℓ-groups).

THEOREM (REDUCTION TO F.S.I. ALGEBRAS).

Consider a variety $\mathsf{V} \subseteq \mathsf{ICRL}$. If $\mathsf{V}_{\mathrm{fsi}}$ locally π-embeds into $\mathsf{K} \subseteq \mathsf{V}$, then

$$\mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{U}\mathbb{C}\mathrm{x}(\mathbb{P}_{\mathrm{fin}}(\mathsf{K})).$$

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Theorem (reduction to f.s.i. algebras).

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Problem: Abelian ℓ-groups are not integral. **Solution:** restrict your $\Box$'s.

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Problem: Abelian ℓ-groups are not integral. **Solution:** restrict your $\square$'s.

A conucleus $\square$ is **negative** $\square x \leq 1$, or equivalently it $\mathbf{A}_{\square}$ is integral.

LEMMA. Totally ordered Abelian ℓ-groups locally π-embed into the **finite lexicographic powers** of $\mathbb{Z}$. These in turn locally π-embed into $\mathbb{P}_{\mathrm{fin}}(\mathbb{Z})$.

Generation for conuclear Abelian ℓ-groups

The universal class of **negatively**

- conuclear Abelian ℓ-groups is generated by $\mathbb{C}\mathrm{x}^{-}(\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}))$.
- $\wedge$-conuclear Abelian ℓ-groups is generated by $\mathbb{C}\mathrm{x}^{-}_{\wedge}(\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}))$.
- conuclear totally ordered Abelian groups is generated by $\mathbb{C}\mathrm{x}^{-}(\mathbb{L}\mathrm{ex}_{\mathrm{fin}}(\mathbb{Z}))$.

Here $\mathbb{L}\mathrm{ex}_{\mathrm{fin}}(\mathbb{Z})$ denotes the class of finite lexicographic powers of $\mathbb{Z}$.

Generation for conuclear MV-algebras

The universal class of

- conuclear MV-algebras has the FEP
- $\wedge$-conuclear MV-algebrass has the FEP.
- conuclear totally ordered MV-algebras **does not** have the FEP.

Decidability

All of the above classes have a decidable universal theory.

The Montagna–Tsinakis representation immediately yields the following.

Decidability for cancellative ICRLs

The following universal classes have a decidable universal theory:

- Cancellative ICRLs.
- Fully distributive cancellative ICRLs.
- Totally ordered cancellative ICRLs.

So do semilinear cancellative ICRLs.

The first case was a long-standing open problem, see Bahls et al. (2003).

The totally ordered and semilinear case were obtained by Horčík (2006).

Horčík (2006) does not **explicitly** discuss conuclei, but his methods would probably have sufficed to handle the totally ordered neg. conuclear case.

Open problem. Decidability for conuclear Abelian ℓ-groups and totally ordered groups. Complexity of the decidable universal theories.

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Remark. π-embeddings reveal an interesting new fragment of first-order logic of ordered algebras, namely **nuclear universal sentences**, which are universal sentences with atomic formulas restricted to $t(x_1, \ldots, x_n) \leq y$.

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Thank you for your attention!