A Model Companion for an Expansion of Abelian Lattice-Ordered Groups

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We recall that for any ℓ-group G and $a \in G$ with $0 \leqslant a$, we have that $[0, a]$ is an MV-algebra, whose operations are given by:

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By Mundici, every MV-algebra arises in this way, and this can be made into a functorial equivalence between MV-algebras and ℓ-groups with a distinguished order unit.

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$$V(a) = \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid a \in G\}.$$

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(E_3) The principal ℓ-ideal generated by $a \in \mathcal{G}$ is equal to the bipolar of a. $(\ell(a) = a^{\perp\perp}$ for all $a \in G.)$

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In particular, $(\Gamma^X, \mathrm{Pow}(X), \{- \geqslant 0\})$ is an example of a valued ℓ-group.

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In particular, $(\Gamma^X, \mathrm{Pow}(X), \{- \geqslant 0\})$ is an example of a valued ℓ-group. We call a valued ℓ-group of this form a **standard structure**.

General ℓ-Groups as Valued ℓ-Groups

Let G be an ℓ-group, $X \subseteq \ell\text{-Spec}(G)$ a dense spectral subspace, and:

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Again, $(G, \bar{\mathcal{K}}(X), V(- \wedge 0) \cap X)$ is an example of a valued ℓ-group.

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Further, we always consider a valued ℓ-group G in the two-sorted language:

$$\mathfrak{L} := \mathcal{L}_{\ell\text{-Grp}} \cup \mathcal{L}_{\text{BDLat}} \cup \{P\}$$

where $\mathcal{L}_{\ell\text{-Grp}} = (+, -, 0, \leqslant, \wedge, \vee)$ and $\mathcal{L}_{\text{BDLat}} = (\sqsubseteq, \sqcap, \sqcup, \top, \bot)$.

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Theorem (Topological Representation Theorem)

Every valued ℓ-group is (up to isomorphism) of the form $(G, \bar{\mathcal{K}}(X), V(- \wedge 0) \cap X)$ *for some dense spectral subspace* $X \subseteq \ell\text{-Spec}(G)$.

This has many applications in the study of valued ℓ-groups, of which we will look at some of the more immediate consequences.

Corollary

Let $V = (G, L, P)$ be a valued ℓ-group with G totally-ordered. Then, L is totally-ordered.

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Recall that a is a weak order unit if and only if $D(a) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid a \notin \mathfrak{p}\}$ is dense.

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Recall that a is a weak order unit if and only if $D(a) := \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid a \notin \mathfrak{p}\}$ is dense.
Thus, if $P(-a) = \perp$, then $X \cap V(a) = \emptyset$, and so $X \subseteq D(a)$. In particular, $D(a)$ is dense, and so we are done. QED

As well as topological representations, we also have functional representations for valued ℓ-groups. In particular, we can represent (G, L, P) as a substructure of a standard structure:

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The Functional Representation Theorem is one of two key ingredients in the characterisation of algebraically closed and existentially closed valued ℓ-groups.

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Definition (Patching)

Let $V = (G, L, P)$ be a valued ℓ-group. Then, V has **patching** if, for all $C, D \in L$ and $f, g \in G$ with $C \wedge D \leqslant P(-|f-g|)$,

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We observe that this is analogous to gluing in the sheaf-theoretic context.

Theorem (Shen-Weispfenning Theorem)

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such that:

$$V \models (\forall \bar{a} : \mathbb{G})(\forall \bar{l} : \mathbb{L})(\phi(\bar{a}, \bar{l}) \leftrightarrow \chi(P(t_1(\bar{a})), \ldots, P(t_k(\bar{a})), \bar{l})).$$

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Further, if $\phi(\bar{v}, \bar{w})$ is (positive) existential, so is $\chi(\bar{p}, \bar{l})$.

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($\Longrightarrow$) This follows from the Functional Representation Theorem, and the fact that $(\Gamma^X, \mathrm{Pow}(X), \{- \geqslant 0\})$ with Γ divisible has these properties.

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With some work, we can also show the following stronger characterisation, similar to an Ax-Kochen-Ershov principle:

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Corollary

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As G has a weak order unit, then we can observe that G is never existentially closed - indeed, if $\nu \in G^+$ is a weak order unit, then there exists no $a \in G^+$ non-zero with $\nu \wedge a = 0$, and so G doesn't satisfy (E_2) $(\forall a)(0 < a \rightarrow (\exists b)((0 < b)$ and $(a \wedge b = 0)))$.

Theorem

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We observe that this is an entirely first-order axiomatisation in the language $\mathfrak{L}$, and so in particular there is an $\mathfrak{L}$-theory of existentially closed valued ℓ-groups. We call this a **model companion** for the theory of valued ℓ-groups.

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Proof.

In both cases, we apply the Shen-Weispfenning theorem to reduce $\mathfrak{L}$-formulae to $\mathcal{L}_{\mathrm{BDLat}}$-formulae, and then use the fact these properties apply to the theory of atomless Boolean algebras. QED

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[2] Fuxing Shen and Volker Weispfenning. "Continuous Functions: ℓ-groups and decision problems". Unpublished. 1987.

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