Completions and Functional Representation of Monadic Ortholattices

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TACL 2026

Co-funded by the European Union

Ministry of Education, Youth and Sports of the Czech Republic

This talk is based on joint work with **John Harding** (New Mexico State University), **Miguel Peinado** (University of Oklahoma), and **Chun-Yu Lin** (Czech Academy of Sciences):

1. Harding, J., McDonald, J., Peinado, M.: Monadic ortholattices: completions and duality. **Algebra Universalis**, 86 (2025)
2. Lin, C.Y., McDonald, J.: Functional monadic ortholattices and locally finite σ-free polyadic ortholattices. **Algebra Universalis**, 87 (2026)

Some history

1. A monadic Boolean algebra is a Boolean algebra B equipped with a closure operator $\nabla : B \to B$, known as a quantifier, whose closed elements for a Boolean subalgebra
2. Halmos introduced monadic Boolean algebras as algebraic models of the classical single-variable predicate calculus.
3. Aside from logic, a motivating example of monadic Boolean algebras arises by considering B^X where X is a non-empty set and B is a complete Boolean algebra. The Boolean operations on B^X are defined pointwise and the associated quantifier is

$$(\nabla f)(x) = \bigvee \{f(x) : x \in X\}$$

4. Recently, Harding introduced monadic ortholattices and quantum monadic algebras as generalizations of monadic Boolean algebras.

Monadic ortholattices

Definition

An *ortholattice* is a bounded lattice $\langle A; \wedge, \vee, 0, 1\rangle$ equipped with an order-inverting period two complementation ${}^\perp : A \to A$.

Definition

A *monadic ortholattice* is an ortholattice A equipped with a closure operator $\nabla : A \to A$, known as a *quantifier*, satisfying $\nabla(\nabla a)^\perp = (\nabla a)^\perp$.

Example

Let M be a von Neumann subalgebra of the bounded operators $B(H)$ of a Hilbert space H. The (self-adjoint) projections $\mathrm{Proj}(M)$ form a complete subalgebra of the ortholattice $\mathrm{Proj}(H)$ of projection operators on H. For any $\pi \in \mathrm{Proj}(H)$, the quantifier $\nabla\pi$ is the least projection in M above π.

Quantifiers arising via approximating subalgebras

Definition

Let A be an ortholattice. A subalgebra $B \subseteq A$ is an *approximating subalgebra* if for every $a \in A$, there exists a greatest element $a^* \in B$ such that $a^* \leq a$ and least element $a^\dagger \in B$ such that $a \leq a^\dagger$.

Note that a^* exists iff $a^\dagger$ exists and that they are related by the equations:

$$a^* = ((a^\perp)^\dagger)^\perp, \;\; a^\dagger = ((a^\perp)^*)^\perp$$

Approximating subalgebras and quantifiers are connected by the following.

Theorem (Harding (2023))

Let A be an ortholattice and let $B \subseteq A$ be an approximating subalgebra. Then $\nabla a = a^\dagger$ and $\Delta a = a^*$ are existential and universal quantifiers on A. Conversely, if $\nabla : A \to A$ is an existential quantifier, then $\nabla[A]$ is an approximating subalgebra of A.

Example

For any element x, ∇x is the least $\bullet$-element above x.

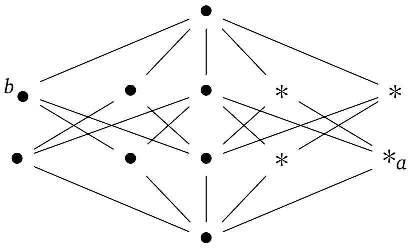

Figure 1: 12-element monadic ortholattice

Remark

Unlike every monadic Boolean algebra, $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$ does not necessarily obtain in monadic ortholattices. In the above Hasse diagram, we have $\nabla(a \wedge \nabla b) = 0$ whereas $\nabla a \wedge \nabla b$ is an atom.

MacNeille completion

Definition

Let A be a bounded lattice and let $\langle e; C\rangle$ be a completion of A. Then:

1. e is $\bigwedge$-*dense* if every $a \in C$ is meet of elements in the e-image of A;
2. e is $\bigvee$-*dense* if every $a \in C$ is join of elements in the e-image of A;
3. $\langle e; C\rangle$ is the MacNeille completion of A if e is $\bigwedge$-dense and $\bigvee$-dense;

We denote the MacNeille completion of A by $\overline{A}$.

Definition

For a monadic ortholattice A, its MacNeille completion is $\langle e; \overline{A}\rangle$ equipped with additional operators $\overline{^{\perp}}, \overline{\nabla} : \overline{A} \to \overline{A}$ defined by

$$a^{\overline{\perp}} = \bigwedge\{e(b^{\perp}) : e(b) \leq a\}, \quad \overline{\nabla} a = \bigvee\{e(\nabla b) : e(b) \leq a\}$$

Monadic Orthoframes

Definition

An *orthoframe* is pair $\langle X, \perp \rangle$ such that X is a set and $\perp \subseteq X^2$ is a binary orthogonality relation which is irreflexive and symmetric. Moreover:

1. For $U \subseteq X$, let $U^{\perp} = \{x \in X : x \perp U\} = \{x \in X : x \perp y \text{ for all } y \in U\}$
2. $U \subseteq X$ is *biorthogonally closed* iff $U = U^{\perp\perp}$. Notation: $B(X)$.

A *monadic orthoframe* is a triple $\langle X; \perp, R \rangle$ such that:

1. $\langle X; \perp \rangle$ is an orthoframe;
2. $R \subseteq X \times X$ is reflexive and transitive;
3. $R[R[\{x\}]^{\perp}] \subseteq R[\{x\}]^{\perp}$

Theorem (Birkhoff (1967), Harding (2023))

If X is a monadic orthoframe, then $\langle B(X); \cap, \sqcup, {}^{\perp}, \emptyset, X, \nabla_R \rangle$ is a complete monadic ortholattice under $U \sqcup V = (U \cup V)^{\perp\perp}$ and $\nabla_R U = R[U]^{\perp\perp}$.

MacLaren frames

Definition

Let A be a monadic ortholattice. The *MacLaren frame* of A is a relational frame $X_A^M = \langle A \setminus \{0\}; \perp_A, R_A \rangle$ where

1. $A \setminus \{0\}$ is the non-zero elements of A;
2. $a \perp_A^M b \Leftrightarrow a \leq b^\perp$;
3. $aR_A^M b \Leftrightarrow b \leq \nabla a$.

Lemma

If A is a monadic ortholattice, then X_A is a monadic orthoframe. Therefore, $B(X_A^M)$ is a complete monadic ortholattice.

MacNeille completion

Lemma

Every monadic ortholattice A can be embedded into $B(X_A^M)$.

Proof idea

We show that $g(a) = \{b \in A \setminus \{0\} : b \leq a\}$ provides the desired embedding.

Theorem

$\langle g, B(X_A^M)\rangle$ *is the MacNeille completion of A.*

Corollary

Monadic ortholattices are closed under MacNeille completions.

Definition

Let A be a bounded lattice and let $\langle e; C\rangle$ be a completion of A. Then:

1. e is *dense* if every element if every $a \in C$ is a meet of joins and join of meets of elements in the e-image of A;
2. e is *compact* if for all $S, T \subseteq A$, there exist finite subsets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge e[S] \leq \bigvee e[T]$ implies $\bigwedge e[S'] \leq \bigvee e[T']$;
3. e is the *canonical completion* of A of e is dense and compact.

We denote the canonical completion by A^σ. $K = \{\text{closed elements of } A^\sigma\}$.

Definition

The *canonical extension* of a monadic ortholattice A is A^σ equipped with additional operators ${}^{\perp^\sigma}, \nabla^\sigma : A^\sigma \to A^\sigma$ defined by

$$a^{\perp^\sigma} = \bigwedge\left\{\bigvee\{e(b^\perp) : c \leq e(b)\} : c \leq a \ \text{ and } \ c \in K\right\}$$

$$\nabla^\sigma a = \bigvee\left\{\bigwedge\{e(\nabla b) : c \leq e(b)\} : c \leq a \ \text{ and } \ c \in K\right\}$$

Goldblatt frames

Definition

Let A be a monadic ortholattice. Its *Goldblatt frame* is a relational structure $X_A^G = \langle \mathfrak{F}(A); \perp_A^G, R_A^G \rangle$ satisfying the following conditions:

1. $\mathfrak{F}(A)$ is the collection of non-empty proper filters of A;
2. $x \perp_A^G y$ iff exists $a \in x$ such that $a^\perp \in y$;
3. $x R_A^G y$ iff $\nabla[x] \subseteq y$ where $\nabla[x] = \{\nabla a : a \in x\}$.

Lemma

If A is a monadic ortholattice, then X_A^G is a monadic orthoframe. Therefore $B(X_A^G)$ is a complete monadic ortholattice.

Canonical extension

Lemma

Every monadic ortholattice A can be embedded into $B(X_A^G)$.

Proof idea

We show that $\phi(x) = \{x \in \mathfrak{F}(A) : a \in x\}$ gives the desired embedding. For injectivity, assume $a \not\leq b$, then clearly $a \in \uparrow a$ but $b \notin \uparrow a$. Therefore $\uparrow a \in \phi(a)$ but $\uparrow a \notin \phi(b)$ so $\phi(a) \not\subseteq \phi(b)$.

Theorem

$\langle \phi ; B(X_A^G) \rangle$ *is the canonical extension of A.*

Corollary

Monadic ortholattices are closed under canonical extensions.

Functional monadic ortholattices

Definition

Let X be a non-empty set and let A be a complete ortholattice. Then the algebra $\langle A^X; \cdot, +, -, c_0, c_1, \nabla \rangle$ defined for all $f, g \in A^X$ by:

1. $(f \cdot g)(x) = f(x) \wedge g(x)$
2. $(f + g)(x) = f(x) \vee g(x)$
3. $-f(x) = f(x)^\perp$
4. $c_0(x) = 0$, $c_1(x) = 1$
5. $(\nabla f)(x) = \bigvee \{f(x) : x \in X\}$

is a monadic ortholattice, known as the *full functional monadic ortholattice* determined by X and A.

Definition

A *functional monadic ortholattice* is a subalgebra of a full functional monadic ortholattice.

Functional representation

Theorem

Every monadic ortholattice can be embedded into a full functional one.

Proof idea

The proof uses tools from the theory of regular completions, directed limits, and super-amalgamations.

Corollary

Every monadic ortholattice is isomorphic to a functional one.

Related results and structures

- Stone duality for monadic ortholattices

  - Harding, J., McDonald, J., Peinado, M.: Monadic ortholattices: completions and duality. **Algebra Universalis**, 86 (2025)

- Polyadic and locally finite diagonal-free cylindric ortholattices

  - Lin, C.Y., McDonald, J.: Functional monadic ortholattices and locally finite σ-free polyadic ortholattices. **Algebra Universalis**, 87 (2026)

- Spectral duality and completions for cylindric ortholattices and BAs

  - McDonald, J.: Canonical completion and duality for cylindric ortholattices and cylindric Boolean algebras. **Studia Logica** (2026)

- Sasaki hook in quantum monadic and quantum cylindric algebras

  - McDonald, J.: Orthogonality relations and operators on bounded quasi-implication algebras. **Order**, 42 (2025)
  - McDonald, J.: Cylindric quasi-implication algebras. Under review at **Journal of Applied Non-Classical Logics**

Open Problems

Question 1

Is every quantum monadic algebra isomorphic to a functional one?

- A special case of the above question may be addressed by considering complete quantum monadic algebras that contain approximating Boolean subalgebras.
- A more fundamental and long-standing open problem is the following.

Question 2

Can every orthomodular lattice be embedded into a complete one?

- Question 2 fails for MacNeille and canonical completions (consult Harding (1991, 1998)).

References

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