Arithmetic polytopes, ordered groups, and affine monoids: a duality theorem

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A joint work with Vincenzo Marra and Luca Reggio

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Outline

Motivation

Integral polyhedral geometry plays a central rôle in Łukasiewicz logic.

- The algebraic counterpart to Łukasiewicz logic is provided by Chang's MV-algebras.
- In particular, the category of finitely presented MV-algebras is dually equivalent to the category of rational polyhedra and piecewise-affine maps with integer coefficients [2, 3].
- The building blocks of the latter category are rational polytopes and affine maps between them whose scalar components, in coordinates, have integer coefficients.

[2] Vincenzo Marra, Luca Spada, *The dual adjunction between MV-algebras and Tychonoff spaces*, *Studia Logica* 100 (2012), no. 1–2, pp. 253–278.

[3] Vincenzo Marra, Luca Spada, *Duality projectivity, and unification in Łukasiewicz logic and MV-algebras*, *Ann. Pure Appl. Logic*, 164(3):192 210,2013.

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An *affine combination* of S is a point of the form

$$\sum_{p \in F} \lambda_p p, \tag{1}$$

for a finite subset $F \subseteq S$ and $\lambda_p \in \mathbb{Q}$ such that $\sum_{p \in F} \lambda_p p = 1$; if $\lambda_p \in \mathbb{Q}_{\geq 0}$ for all $p \in F$, then (1) is a *convex combination* of S.

The *convex hull* of S is the subset of $\mathbb{Q}^n$ consisting of all convex combinations of S.

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A point v of a polytope P such that $v \notin \mathrm{conv}(P - \{v\})$ is a *vertex* of P.

$\mathbb{Z}$-maps

An affine map $\mathbb{Q}^n \to \mathbb{Q}^m$ is $\mathbb{Z}$-*affine* if it descends to a function $\mathbb{Z}^n \to \mathbb{Z}^m$.

For subsets $X \subseteq \mathbb{Q}^n$, $Y \subseteq \mathbb{Q}^m$, a map $X \to Y$ is $\mathbb{Z}$-*affine* if it can be extended to a $\mathbb{Z}$-affine map $\mathbb{Q}^n \to \mathbb{Q}^m$.

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Definition

The *category of arithmetic polytopes* has polytopes as objects and $\mathbb{Z}$-maps as arrows.

Preordered groups

Definition

A *preordered group* is an Abelian group $(A, +, 0)$ endowed with a preorder relation $\preccurlyeq$ such that for all $x, y, z \in A$

$$x \preccurlyeq y \quad \textit{implies} \quad x + z \preccurlyeq y + z.$$

If $\preccurlyeq$ is antisymmetric, then A is a *(partially) ordered group*; in this case we write $\leqslant$ for $\preccurlyeq$.

An element $a \in A$ is *positive* if $0 \preccurlyeq a$. The *positive cone* of A, denoted A^+, is the set of positive elements of A.

A *(strong order) unit* in a preordered group A is an element $1 \in A^+$ such that for all $a \in A$ there exists $n \in \mathbb{N}$ such that $a \preccurlyeq n1$. A preordered group endowed with a unit is a *unital group*.

Categories of preordered groups

Definition

The category $\mathsf{A}_{\preccurlyeq}$ has preordered groups as objects and monotone group morphisms as arrows.

The category $\mathsf{A}_{\leq}$ is the full subcategory of $\mathsf{A}_{\preccurlyeq}$ whose objects are ordered groups.

The category $\mathsf{A}^1_{\preccurlyeq}$ has unital groups as objects and unital morphisms (i.e. monotone group morphisms that preserve units) as arrows.

The category $\mathsf{A}^1_{\leq}$ is the full subcategory of $\mathsf{A}^1_{\preccurlyeq}$ whose objects are unital groups whose preorder is an order.

Proposition

Let $f\colon A \to B$ an arrow in $\mathsf{A}^1_{\preccurlyeq}$ or $\mathsf{A}^1_{\leq}$. Then the following hold:

1. f is a monomorphism if, and only if, f is injective.
2. f is a regular monomorphism if, and only if, f is a monomoprhism and $A^+ = f^{-1}B^+$.
3. f is an epimorphism if, and only if, f is surjective.
4. f is a regular epimorphism if, and only if, f is an epimorphism and $fA^+ = B^+$.

This proposition specializes to the setting of unital groups a result originally proved for non-Abelian ordered groups in M.M. Clementino, N. Martins-Ferreira, A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra 223 (10) (2019) 4226-4245. ([6])

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A monoid M is *strict* if $x + y = 0$ implies $x = 0$ for all $x, y \in M$.

- $(A, \preccurlyeq)$ is an order if, and only if, A^+ is strict.

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A group morphism $f : A \to B$ between preordered groups is monotone if, and only if, $fA^+ \subseteq B^+$.

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- $(\mathrm{Aff}_{\mathbb{Z}}(P,\mathbb{Q}),\leq^{\cdot})$ is unital with $1_P\colon P\to\mathbb{Q}$ as unit.

Thus, we can define the following contravariant functor:

$$\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})\colon \mathsf{P}_{\mathbb{Z}} \dashrightarrow \mathsf{A}^1_{\leq}$$

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f"'] & \mathrm{Aff}_{\mathbb{Z}}(P,\mathbb{Q}) \\
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Let $\leq$ be an order on the set X. We say that the order $\leq$ is *directed* if for all $x, y \in X$ there exists $z \in X$ such that $x \leq z$ and $y \leq z$.

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Theorem

Let $P \subseteq \mathbb{Q}^n$ be a polytope. The unital group $\mathrm{Aff}_{\mathbb{Z}}(P, \mathbb{Q})$ is free, the order is directed and the positive cone $\mathrm{Aff}_{\mathbb{Z}}(P, \mathbb{Q})^+$ is affine and unperforated.

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Definition

A *polytopal group* is any order-reflecting subobject in $\mathsf{A}^1_\leq$ of a simplicial group. We write $\mathsf{G}^1_\leq$ for the full subcategory of $\mathsf{A}^1_\leq$ whose objects are polytopal groups.

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Theorem

A unital group is polytopal if, and only if, it is free, the order is directed and A^+ is affine and unperforated.

From polytopes to polytopal groups

We can therefore consider the functor $\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})$ restricted to $\mathsf{G}^1_{\leq}$:

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- $\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})$ is not representable by $\mathbb{Q}$.

From polytopal groups to polytopes

A *(rational) state* of polytopal group A is a morphism $A \to \mathbb{Q}$ in $\mathsf{A}^1_{\leq}$. We denote with $\mathrm{St}(A, \mathbb{Q})$ the set of states of A.

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For $k \in \mathbb{N}$, order the Cartesian product $\mathbb{N}^k$ coordinatewise with respect to the usual order of $\mathbb{N}$. Any strictly descending chain in $\mathbb{N}^k$ is finite. Also, any antichain in $\mathbb{N}^k$ is finite.

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- $\mathrm{St}(A, \mathbb{Q})$ is a polytope in $\mathbb{Q}^{\alpha A}$.

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Hence we can define the following contravariant functor:

$$\mathrm{St}(-,\mathbb{Q})\colon \mathsf{G}^1_{\le} \dashrightarrow \mathsf{P}_{\mathbb{Z}}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "h"'] & \mathrm{St}(A,\mathbb{Q}) \\
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Duality

Duality theorem

The following functors provide a duality:

$$\begin{tikzcd} \mathsf{P}_{\mathbb{Z}} \arrow[rr, bend left, dashed, "\mathrm{Aff}_{\mathbb{Z}}(-{,}\mathbb{Q})"] & & \mathsf{G}^1_{\leq} \arrow[ll, bend left, dashed, "\mathrm{St}(-{,}\mathbb{Q})"] \end{tikzcd}$$

Units of the duality:

Let P be a polytope,

$$\eta_P \colon P \to \mathrm{St}(\mathrm{Aff}_{\mathbb{Z}}(P, \mathbb{Q}), \mathbb{Q}),\ v \mapsto ev_v$$

where the *evaluation map at* v, ev_v, is the state $\mathrm{Aff}_{\mathbb{Z}}(P, \mathbb{Q}) \to \mathbb{Q}$, $f \mapsto fv$.

Let A be a polytopal group,

$$\epsilon_A \colon A \to \mathrm{Aff}_{\mathbb{Z}}(\mathrm{St}(A, \mathbb{Q}), \mathbb{Q}),\ x \mapsto ev_x,$$

where the *evaluation at* x, ev_x, is the $\mathbb{Z}$-map $\mathrm{St}(A, \mathbb{Q}) \to \mathbb{Q}$, $s \mapsto sx$.

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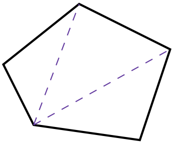
Let P be a polytope and let $\{T_i\}_{i \in I}$ be a triangulation by diagonals of P.



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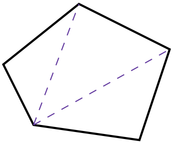

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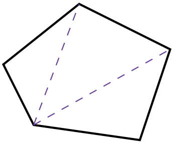

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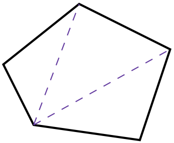

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- $\mathrm{St}(-, \mathbb{Q})$ sends finite connected limits to finite connected colimits.

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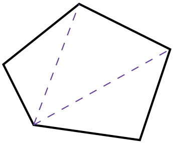
- η_T is an isomorphism for T simplex.

Let P be a polytope and let $\{T_i\}_{i \in I}$ be a triangulation by diagonals of P.



- Let I be the category that has the simplexes T_i's and all their faces as objects and the inclusion maps as arrows: $P \cong \operatorname{colim} T_i$.
- $\mathrm{Aff}_{\mathbb{Z}}(-, \mathbb{Q})$ sends finite connected colimits to finite connected limits.
- $\mathrm{St}(-, \mathbb{Q})$ sends finite connected limits to finite connected colimits.

η_P is an isomorphism for any polytope P.

Unit ϵ

Let A be a polytopal group. In order to prove that ϵ_A is an isomorphism, we need to prove that it is a group isomorphism which is surjective on the positive cones.

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Proposition

Let A be a polytopal group, and let $x \in A$. If $sx \geq 0$ for all states s, then $x \in A^+$.

- ϵ_A is surjective on the positive cones.

Pointed affine monoids

Let $\mathsf{M}_{\mathsf{aff}}$ be the category of affine monoids.

- The positive cones of the polytopal groups are affine monoids.
- We want to find a way to select an element in such a way that it behaves like a unit.

For an affine monoid M, let $\mathrm{H}(M)$ be the Hilbert basis of M. Let $h\colon \mathbb{N} \to M$ a monoid morphism and let $h1 = \sum_{x \in \mathrm{H}(M)} n_x x$. We say that h is *strongly pointed* if $n_x \neq 0$ for all $x \in \mathrm{H}(M)$.

Let $\mathbb{N}/\mathsf{M}^*_{\mathsf{aff}}$ be the full subcategory of the coslice category $\mathbb{N}/\mathsf{M}_{\mathsf{aff}}$ whose objects are strongly pointed morphisms.

Conjecture

$\mathsf{G}^1_{\leq} \cong \mathbb{N}/\mathsf{M}^*_{\mathsf{aff}}$.

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