Quantale-enriched proof theory

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Motivation

- Fuzzy (modal) logics:
  - $\varphi ::= p \mid r \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \cdot \varphi \mid \varphi \rhd \varphi \mid \Box\varphi \mid \Diamond\varphi$
  - Many-valued sets and many-valued relations
  - (hyper)sequent calculi interpreted on MV-algebras (with normal operators)

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- (non-distributive) Lattice based modal logics:
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- $\varphi \vdash_s \psi$
  - Confidence, distance, cost, ...

A long time ago in a galaxy far,
far away....

QUANTALE-ENRICHED PROOF THEORY

Derivability was a binary *relation:* a conclusion either *followed* from its premises, *or it did not.*

Today, we travel to the *Quantale*-Enriched Galaxy, *where derivability* measures *not only* existence, but also *distance*, cost, confidence, *or proximity* to the normal form.

Quantale-enriched categories

A quantale is a complete semilattice with a monoidal structure $(\Omega, \sqsubseteq, \bigsqcup, e, \cdot)$ such that

$$s \sqsubseteq r \rhd t \quad \Leftrightarrow \quad r \cdot s \sqsubseteq t \quad \Leftrightarrow \quad r \sqsubseteq t \lhd s$$

An Ω-enriched category consists of a set X and a function $X(-,-) : X \times X \longrightarrow \Omega$ such that

$$e \sqsubseteq X(x,x) \quad \text{and} \quad X(x,y) \cdot X(y,z) \sqsubseteq X(x,z)$$

for all $x, y, z \in X$. Quantale-enriched categories are ordered by $x \leqslant_X x' \Leftrightarrow e \sqsubseteq X(x,x')$.
A **functor** $F : X \longrightarrow Y$ between Ω-enriched categories is a function on the underlying sets satisfying

$$X(x,x') \sqsubseteq Y(Fx, Fx').$$

…upsets, downsets,…

The elements of $\mathcal{D}X$ (presheaves) are functors $\varphi : X \to \Omega^{op}$ ($\Omega^{op}(\omega_1, \omega_2) = \omega_1 \lhd \omega_2$) with homs

$$\mathcal{D}X(\varphi', \varphi) = \varphi' \blacktriangleright \varphi = \bigsqcap_{x \in X} (\varphi' x \rhd \varphi x)$$

The elements of $\mathcal{U}X$ (co-presheaves) are functors $\psi : X \to \Omega$ ($\Omega(\omega_1, \omega_2) = \omega_1 \rhd \omega_2$) with homs

$$\mathcal{U}X(\psi, \psi') = \psi \blacktriangleleft \psi' = \bigsqcap_{x \in X} (\psi x \lhd \psi' x)$$

...and (co)limits

Given $G : D \longrightarrow B$, $\varphi \in \mathcal{D}D$, $\psi \in \mathcal{U}D$:
The weighted colimit $\operatorname{colim}_\varphi G$ is the solution of

$$B(\operatorname{colim}_\varphi G, b) = \varphi \blacktriangleright B(G, b)$$

The weighted limit $\lim_\psi G$ is the solution of

$$B(b, \lim_\psi G) = B(b, G) \blacktriangleleft \psi$$

All colimits and limits can be constructed with the special (co)limits where

- D is discrete, and φ, ψ are constant e, which we call joins and meets respectively.
- D is a singleton, which we denote $\langle r \rangle b$ and $[r]b$ and call tensor and power respectively.

Proof theory

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle r \rangle \varphi \mid [r]\varphi \mid \Box\varphi \mid \Diamond\varphi$$

$$\frac{}{p \vdash_e p} \qquad \frac{}{p \vdash_\bot q} \qquad \frac{\varphi \vdash_r \chi \qquad \chi \vdash_s \psi}{\varphi \vdash_{r \cdot s} \psi}\ \text{Cut}$$

$$\frac{\varphi \vdash_s \psi}{\varphi \vdash_s \psi \vee \chi} \qquad \frac{\varphi \vdash_s \psi}{\varphi \vdash_s \chi \vee \psi} \qquad \frac{\varphi \vdash_s \psi \qquad \chi \vdash_s \psi}{\varphi \vee \chi \vdash_s \psi}$$

$$\frac{\varphi \vdash_s \psi}{\Diamond\varphi \vdash_s \Diamond\psi} \qquad \frac{\Diamond\varphi \vdash_s \psi}{\varphi \vdash_s \blacksquare\psi} \qquad \frac{\varphi \vdash_s \blacksquare\psi}{\Diamond\varphi \vdash_s \psi}$$

Proof Theory

$$\frac{\varphi \vdash_s \psi}{\langle s \rangle \varphi \vdash_r \langle r \rangle \psi} \qquad \frac{\varphi \vdash_s \psi}{[r]\varphi \vdash_r [s]\psi}$$

$$\frac{\langle r \rangle \varphi \vdash_s \psi}{\varphi \vdash_{r \cdot s} \psi} \qquad \frac{\varphi \vdash_{r \cdot s} \psi}{\langle r \rangle \varphi \vdash_s \psi} \qquad \frac{\varphi \vdash_s [r]\psi}{\varphi \vdash_{s \cdot r} \psi} \qquad \frac{\varphi \vdash_{s \cdot r} \psi}{\varphi \vdash_s [r]\psi}$$

$$\frac{\varphi \vdash_r \psi}{\varphi \vdash_s \psi} \text{(if } s \sqsubseteq r\text{)} \qquad \frac{\left(\varphi \vdash_r \psi\right)_{r \in I}}{\varphi \vdash_{\bigvee_{r \in I} r} \psi} \vdash\bigvee$$

Label discourse: monoidal structure

$$\mathrm{I}_{e1}\ \frac{\varphi \vdash_{r} \psi}{\varphi \vdash_{e\cdot r} \psi} \qquad \frac{\varphi \vdash_{r} \psi}{\varphi \vdash_{r\cdot e} \psi}\ \mathrm{I}_{e2} \qquad \mathrm{E}_{e1}\ \frac{\varphi \vdash_{e\cdot r} \psi}{\varphi \vdash_{r} \psi} \qquad \frac{\varphi \vdash_{r\cdot e} \psi}{\varphi \vdash_{r} \psi}\ \mathrm{I}_{e2}$$

$$A_{\cdot 1}\ \frac{\varphi \vdash_{(r\cdot s)\cdot t} \psi}{\varphi \vdash_{r\cdot (s\cdot t)} \psi} \qquad \frac{\varphi \vdash_{r\cdot (s\cdot t)} \psi}{\varphi \vdash_{(r\cdot s)\cdot t} \psi}\ A_{\cdot 2}$$

Label discourse: lattice structure

$$\frac{\varphi \vdash_{r} \psi}{\varphi \vdash_{r \sqcap s} \psi}\ \mathrm{I}_{\sqcap 1} \qquad \frac{\varphi \vdash_{s} \psi}{\varphi \vdash_{r \sqcap s} \psi}\ \mathrm{I}_{\sqcap 2} \qquad\qquad \frac{\varphi \vdash_{r \sqcup s} \psi}{\varphi \vdash_{r} \psi}\ \mathrm{E}_{\sqcup 1} \qquad \frac{\varphi \vdash_{r \sqcup s} \psi}{\varphi \vdash_{s} \psi}\ \mathrm{E}_{\sqcup 2}$$

$$C_{\sqcap}\ \frac{\varphi \vdash_{r \sqcap r} \psi}{\varphi \vdash_{r} \psi} \qquad \frac{\varphi \vdash_{r \sqcup r} \psi}{\varphi \vdash_{r} \psi}\ C_{\sqcap} \qquad\qquad E_{\sqcap}\ \frac{\varphi \vdash_{r \sqcap s} \psi}{\varphi \vdash_{s \sqcap r} \psi} \qquad \frac{\varphi \vdash_{r \sqcup s} \psi}{\varphi \vdash_{s \sqcup r} \psi}\ E_{\sqcap}$$

$$A_{\sqcap 1}\ \frac{\varphi \vdash_{(r \sqcap s) \sqcap t} \psi}{\varphi \vdash_{r \sqcap (s \sqcap t)} \psi} \qquad \frac{\varphi \vdash_{r \sqcap (s \sqcap t)} \psi}{\varphi \vdash_{(r \sqcap s) \sqcap t} \psi}\ A_{\sqcap 2} \qquad A_{\sqcup 1}\ \frac{\varphi \vdash_{(r \sqcup s) \sqcup t} \psi}{\varphi \vdash_{r \sqcup (s \sqcup t)} \psi} \qquad \frac{\varphi \vdash_{r \sqcup (s \sqcup t)} \psi}{\varphi \vdash_{(r \sqcup s) \sqcup t} \psi}\ A_{\sqcap 2}$$

Label discourse: meet elimination

$$\dfrac{\varphi \vdash_{r \sqcap s} \psi \qquad \begin{array}{c} \varphi \vdash_r \psi \\ \vdots \\ \varphi \vdash_t \psi \end{array} \qquad \begin{array}{c} \varphi \vdash_s \psi \\ \vdots \\ \varphi \vdash_t \psi \end{array}}{\varphi \vdash_t \psi}\ \mathsf{E}_{\sqcap}$$

Cut elimination

The cut-elimination metatheorem for display calculi can be generalized to quantale-enriched display calculi:

Theorem

If $\varphi \vdash_r \psi$ is derivable in Ω-DL, then there exists some $s \in \Omega$ such that $r \sqsubseteq s$ and $\varphi \vdash_s \psi$ is derivable in Ω-cfDL.

- The proof follows the standard argument of pushing the cut towards leafs and towards subformulas.
- Pushing the cut upwards increases the value of the label in the sequent.
- New reading: The label also encodes the distance from the normal form (cut free) proof.

A conjecture

Conjecture

The rule

$$\frac{(\varphi \vdash_r \psi)_{r \in I}}{\varphi \vdash_{\bigvee_{r \in I} r} \psi} \vdash_{\bigvee}$$

is admissible. In particular, if $\varphi \vdash_r \psi$ and $\varphi \vdash_s \psi$ are derivable in Ω-DL$- \vdash_{\bigvee}$ then $s \sqsubseteq r$ or $r \sqsubseteq s$.

Semantics

A Ω-polarity is a tuple $\mathbb{P} = (X, A, \mathbf{I})$, such that X and A are sets and $\mathbf{I} : X \times A \to \Omega$, is a Ω-relation between A and X.

We interpret formulas $[\![-]\!] : \mathrm{Fm} \to (X \to \Omega)$ and $(\![-]\!) : \mathrm{Fm} \to (A \to \Omega)$ as:

$$
\begin{array}{llll}
[\![\top]\!](x) & = \top & (\![\top]\!)(a) & = \bigsqcap_{x \in X}(\top \rhd \mathbf{I}(x,a)) \\
(\![\bot]\!)(a) & = \top & [\![\bot]\!](x) & = \bigsqcap_{a \in A}(\mathbf{I}(x,a) \lhd \top) \\
[\![\varphi \wedge \psi]\!](x) & = [\![\varphi]\!](x) \sqcap [\![\psi]\!](x) & (\![\varphi \wedge \psi]\!)(a) & = \bigsqcap_{x \in X}([\![\varphi \wedge \psi]\!](x) \rhd \mathbf{I}(x,a)) \\
(\![\varphi \vee \psi]\!)(a) & = (\![\varphi]\!)(a) \sqcap (\![\psi]\!)(a) & [\![\varphi \vee \psi]\!](x) & = \bigsqcap_{a \in A}(\mathbf{I}(x,a) \lhd (\![\varphi \vee \psi]\!)(a)) \\
[\![[r]\varphi]\!](x) & = [\![\varphi]\!](x) \lhd r & (\![[r]\varphi]\!)(a) & = \bigsqcap_{x \in X}([\![[r]\varphi]\!](x) \rhd \mathbf{I}(x,a)) \\
(\![\langle r\rangle\varphi]\!)(a) & = r \rhd (\![\varphi]\!)(a) & [\![\langle r\rangle\varphi]\!](x) & = \bigsqcap_{a \in A}(\mathbf{I}(x,a) \lhd (\![\langle r\rangle\varphi]\!)(a))
\end{array}
$$

We interpret sequents as:

$$
\mathbb{P} \models \varphi \vdash_r \psi \quad \text{iff} \quad r \sqsubseteq \bigsqcap_{x \in X}\left([\![\varphi]\!]_{\mathbb{P}}(x) \rhd [\![\psi]\!]_{\mathbb{P}}(x)\right) \quad \text{iff} \quad r \sqsubseteq \bigsqcap_{a \in A}\left((\![\varphi]\!)_{\mathbb{P}}(a) \lhd (\![\psi]\!)_{\mathbb{P}}(a)\right).
$$

Completeness

- What is completeness for a sequential calculus?

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Theorem (Lindenbaum-Tarski)

The sequent $\varphi \vdash_r \psi$ is derivable in Ω-DL if and only if for every Ω-polarity, $\mathbb{P}$, $\mathbb{P} \models \varphi \vdash_r \psi$.

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- Filters and ideals correspond to limit preserving co-presheaves and co-limit preserving presheaves.
- Modal operators?

Canonical extension

$$\begin{tikzcd}
\mathcal{D}F \arrow[rr, bend left, "- \blacktriangleright \mathrm{I}"] \arrow[rr, phantom, "\perp"] \arrow[dr, dashed, bend right] \arrow[dr, phantom, "\dashv"] & & \mathcal{U}I \arrow[ll, bend left, "\mathrm{I} \blacktriangleleft -"] \arrow[dl, dashed, bend left] \arrow[dl, phantom, "\dashv"'] \\
& C^{\delta} \arrow[ul, hook] \arrow[ur, hook] & \\
F \arrow[uu] \arrow[ur, "f \mapsto \overline{f}"] \arrow[rr, dotted, "\mathrm{I}"'] & & I \arrow[uu] \arrow[ul, "i \mapsto \overline{i}"'] \\
& C \arrow[ul] \arrow[ur] \arrow[uu, bend right, "{[-]}"'] &
\end{tikzcd}$$

Burning questions

- Display calculi interpreted on quantale-enriched categories.
- Cut-elimination & completeness.
- Axiomatic extensions?
- Algebraic proof theory?
- Correspondence and display?

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Thank you!