Evolution Systems and Embedding-Projection Pairs

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W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Ann. Pure Appl. Logic 165 (2014) 1755–1811

M. Droste, R. Göbel, *A categorical theorem on universal objects and its application in abelian group theory and computer science*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), 49–74, Contemp. Math., 131, Part 3, Amer. Math. Soc., Providence, RI, 1992

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An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.

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A path is a finite composition of transitions.

Finite objects

Definition

An object X is finite if there exists a path from the origin Θ to X.

Confluence

Definition

We say that $\mathcal{E}$ is confluent if for every finite object Z, for every **paths** $f\colon Z \to X$, $g\colon Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

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We say that $\mathcal{E}$ is locally confluent if for every finite object Z, for every **transitions** $f\colon Z \to X$, $g\colon Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

Amalgamation

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We say that $\mathcal{E}$ has the finite transition amalgamation property (FTAP) if for every finite object Z, for every transitions $f\colon Z \to X$, $g\colon Z \to Y$ there exist transitions f', g' such that $f' \circ f = g' \circ g$.

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Proposition

Finite transition amalgamation property implies confluence.

Theorem

Assume $\mathcal{E}$ is an essentially countable evolution system with the FTAP. Then:

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Fraïssé theory revisited

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$$\begin{tikzcd}
\Theta = U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_n \arrow[r] \arrow[d] & \cdots \arrow[r] & U_m \arrow[r] & \cdots \\
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\Theta = U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_n \arrow[r] \arrow[d, red] & \cdots \arrow[r] & U_m \arrow[r] & \cdots \\
 & & & Y \arrow[urr, dashed, blue] & & &
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Classical Fraïssé limits

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Let $\mathcal{F}$ be a class of finite structures in a fixed language $\mathcal{L}$ consisting of finitely many relations. Assume $\mathcal{F}$ is closed under isomorphisms and substructures.

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We can convert $\mathcal{F}$ to an evolution system, where the origin is $\emptyset$ and transitions are one-point extensions. The category $\mathfrak{V}$ consists of all structures in the language $\mathcal{L}$, where the arrows are all homomorphisms. Assume $\mathcal{F}$ has the amalgamation property. Then the generic evolution leads to the Fraïssé limit of $\mathcal{F}$, the unique homogeneous countable structure U_∞ whose age is $\mathcal{F}$.

The amalgamation property

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$$\begin{tikzcd} A \arrow[r, "e_0"] \arrow[d, "e_1"'] & B_0 \arrow[d, dashed, "e_0'"] \\ B_1 \arrow[r, dashed, "e_1'"'] & C \end{tikzcd}$$

Embedding-projection pairs

Definition

Fix a category $\mathfrak{V}$. The category of EP pairs over $\mathfrak{V}$, denoted by $\ddagger\mathfrak{V}$, consists of the same objects as $\mathfrak{V}$, while arrows are pairs of the form $\langle e, p \rangle$, where $p \circ e$ is the identity.

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Definition

Given an evolution system $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta\rangle$, we define the system $\ddagger\mathcal{E} := \langle \ddagger\mathfrak{V}, \ddagger\mathcal{T}, \Theta\rangle$, where

$$\ddagger\mathcal{T} := \{\langle e, p\rangle \in \ddagger\mathfrak{V} \colon e \in \mathcal{T}\}.$$

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Definition

A (separable) Banach space E has a monotone Schauder basis if there exists a sequence of norm one projections $P_n \colon E \to E_n$ converging pointwise to the identity and such that

1. $P_n \circ P_m = P_m \circ P_n = P_n$ for every $n \leq m$;
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Definition

Let $\mathcal{E} = \langle \mathfrak{B}_1, \mathcal{T}, \Theta \rangle$ be the canonical evolution system of Banach spaces. That is, $\mathfrak{B}_1$ is the category of all Banach spaces with non-expansive linear operators, Θ is the trivial space, and $\mathcal{T}$ consists of linear isometric embeddings $e \colon F \to F'$ such that $\dim(F'/_{e[F]}) \le 1$.

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Claim

Banach spaces with monotone Schauder bases correspond to colimits of evolutions in $\ddagger\mathcal{E}$.

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Let $\mathcal{F}$ be a hereditary class of relational structures, closed under isomorphisms and unions of chains.
We define $\mathcal{E}_{\mathcal{F}} = \langle \mathfrak{V}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}}, \Theta \rangle$, where $\mathfrak{V}_{\mathcal{F}}$ is the category of all homomorphisms between the structures of $\mathcal{F}$, Θ is the trivial structure, and $\mathcal{T}_{\mathcal{F}}$ consists of all embeddings "adding" at most one element.

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It is natural to study the system $\ddagger\mathfrak{V}_{\mathcal{F}}$. Note that only left-invertible transitions survive when passing to the system of EP pairs. In particular, the "trivial" structure should should not be empty.

Fraïssé theory for EP pairs

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Graphs

Definition

A natural graph is a graph G whose set of vertices is an initial segment of ω (either ω or a natural number) and for each vertex $n > 0$ of G there exists a homomorphism $r \colon n+1 \to n$ that is the identity on n.

We use the set-theoretic convention: $n = \{0, \dots, n-1\}$.

Theorem (Geschke, Głąb, K.)

There exists a unique natural graph $\mathbb{W}$ with the following extension property:

(E) *Given a finite induced subgraph $A \subseteq \mathbb{W}$, given its one-point extension $B = A \cup \{v\}$ and a projection $p\colon B \to A$, there exists $w \in \mathbb{W}$ with $w > \max A$, such that $A \cup \{w\}$ is isomorphic to B over A and $p(v)$ is the image of w under some projection of $\mathbb{W}$ onto A.*

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In other words, every one-point extension of a finite induced subgraph, in the sense of embedding-projection pairs, is realized in $\mathbb{W}$. Furthermore:

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Theorem

Every natural graph can be embedded into $\mathbb{W}$ in such a way that there is a projection onto its image.

- W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, [arXiv:2109.12600]
- S. Geschke, S. Głąb, W. Kubiś, *Evolutions of finite graphs*, preprint [arXiv:2209.14626]

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Further research:

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Further research:

- Profinite structures (including domains); probability spaces; etc.

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