How to build categorical semantics of proofs for the basic modal Lambek logic and beyond

Giuseppe Greco[1]

Ongoing joint work with:
Brandon A. Laing[1] Peter Faul[2] Fei Liang[3] Apostolos Tzimoulis[4]

[1]Vrije Universiteit Amsterdam, Netherlands
[2]Stellenbosch University, South Africa
[3]Shandong University, China
[4]Université du Luxembourg, Luxembourg

29 July
– TACL 2026 –
Jagiellonian University in Kraków, Poland

The identity of proofs problem

Identifying or telling apart proofs has far-reaching consequences.

- **Philosophy and mathematics**: when do two proofs correspond to the same argument?
- **Computer science**: when do two algorithms correspond to the same program?
- **Linguistics**: how to capture different readings of the same sentence? [more on this later]
- ...

Curry–Howard: proofs $\leftrightarrow$ programs $\leftrightarrow$ morphisms.

Looking for:

1. uniform and modular **structural proof theory**
2. uniform and modular **categorical semantics**

Our results 1/4

- We introduced the class of **proper labelled display calculi**:

sequent labels encode proofs

  - **cut-elimination meta-theorem** à la Belnap [1] is preserved;
  - today: focus on the modal Lambek logic $D^{\ell}.NL_{\diamond}$, but it generalizes to any basic residuated logic.

The language of $D^\ell.NL_\diamond$

$D^\ell.NL_\diamond$ manipulates sequents $\Gamma \xrightarrow{\ell} \Delta$, such that Γ and Δ are *structure metavariables*:

$$\begin{array}{rcl} \mathrm{Fm} \ni A & ::= & p \mid \diamond A \mid \blacksquare A \mid A \otimes A \mid A / A \mid A \backslash A \\ \mathrm{Str}_{\mathcal{F}} \ni \Gamma & ::= & A \mid \hat{\diamond}\Gamma \mid \Gamma \,\hat{\otimes}\, \Gamma \\ \mathrm{Str}_{\mathcal{G}} \ni \Delta & ::= & A \mid \check{\blacksquare}\Delta \mid \Delta \check{/} \Gamma \mid \Gamma \check{\backslash} \Delta \end{array}$$

and ℓ is a *sequent label metavariable*:

$$\begin{array}{rcl} \mathrm{Lb} \ni \ell & ::= & 1_A \mid \ell \circ \ell \mid \diamond\ell \mid \blacksquare\ell \mid \ell \otimes \ell \mid \ell/\ell \mid \ell\backslash\ell \mid \\ & & 1_\diamond(\ell) \mid 1_\blacksquare(\ell) \mid 1_\otimes(\ell) \mid 1_/(\ell) \mid 1_\backslash(\ell) \mid \\ & & \mu(t) \mid \mu^{-1}(\ell) \mid \beta(t) \mid \beta^{-1}(\ell) \mid \gamma(\ell) \mid \gamma^{-1}(\ell) \end{array}$$

The labelled display calculus $D^{\ell}.NL_{\diamond}$ 1/2

Identity and Cut

$$1_A \; \frac{}{A \xrightarrow{1_A} A} \qquad \frac{\Gamma \xrightarrow{k} A \qquad A \xrightarrow{\ell} \Delta}{\Gamma \xrightarrow{\ell \circ k} \Delta} \; \circ_A$$

Residuation and adjunction

$$\beta^{-1} \; \frac{\Gamma \xrightarrow{k} \Delta \check{/} \Pi}{\Gamma \hat{\otimes} \Pi \xrightarrow{\beta^{-1}(k)} \Delta} \qquad \frac{\Gamma \hat{\otimes} \Pi \xrightarrow{k} \Delta}{\Gamma \xrightarrow{\beta(k)} \Delta \check{/} \Pi} \; \beta \qquad \gamma^{-1} \; \frac{\Pi \xrightarrow{k} \Gamma \check{\backslash} \Delta}{\Gamma \hat{\otimes} \Pi \xrightarrow{\gamma^{-1}(k)} \Delta} \qquad \frac{\Gamma \hat{\otimes} \Pi \xrightarrow{k} \Delta}{\Pi \xrightarrow{\gamma(k)} \Gamma \check{\backslash} \Delta} \; \gamma$$

$$\mu^{-1} \; \frac{\Gamma \xrightarrow{k} \check{\blacksquare} \Delta}{\hat{\diamond} \Gamma \xrightarrow{\mu^{-1}(k)} \Delta} \qquad \frac{\hat{\diamond} \Gamma \xrightarrow{k} \Delta}{\Gamma \xrightarrow{\mu(k)} \check{\blacksquare} \Delta} \; \mu$$

The labelled calculus $D^\ell.NL_\Diamond$ 2/2

Tonicity rules

$$/_L \; \frac{A \xrightarrow{k} \Delta \qquad \Gamma \xrightarrow{\ell} B}{A/B \xrightarrow{k/\ell} \Delta \check{/} \Gamma} \qquad \frac{\Gamma \xrightarrow{k} A \qquad \Pi \xrightarrow{\ell} B}{\Gamma \hat{\otimes} \Pi \xrightarrow{k \otimes \ell} A \otimes B} \; \otimes_R \qquad \backslash_L \; \frac{\Gamma \xrightarrow{k} A \qquad B \xrightarrow{\ell} \Delta}{A \backslash B \xrightarrow{k \backslash \ell} \Gamma \check{\backslash} \Delta}$$

$$\blacksquare_L \; \frac{A \xrightarrow{k} \Delta}{\blacksquare A \xrightarrow{\blacksquare k} \check{\blacksquare} \Delta} \qquad \frac{\Gamma \xrightarrow{k} A}{\hat{\Diamond} \Gamma \xrightarrow{\Diamond k} \Diamond A} \; \Diamond_R$$

Translation rules

$$\frac{\Gamma \xrightarrow{k} A \check{/} B}{\Gamma \xrightarrow{1_{/}(k)} A/B} \; /_R \qquad \otimes_L \; \frac{A \hat{\otimes} B \xrightarrow{k} \Delta}{A \otimes B \xrightarrow{1_{\otimes}(k)} \Delta} \qquad \frac{\Gamma \xrightarrow{k} A \check{\backslash} B}{\Gamma \xrightarrow{1_{\backslash}(k)} A \backslash B} \; \backslash_R$$

$$\Diamond_L \; \frac{\hat{\Diamond} A \xrightarrow{k} \Delta}{\Diamond A \xrightarrow{1_{\Diamond}(k)} \Delta} \qquad \frac{\Gamma \xrightarrow{k} \check{\blacksquare} A}{\Gamma \xrightarrow{1_{\blacksquare}(k)} \blacksquare A} \; \blacksquare_R$$

Our results 2/4

- Following Lambek [9], we generalized the **Lindenbaum–Tarski construction** to the categorical level:

objects are formulas
and
morphisms are equivalence classes of proofs

  - the free cat. F generated by $D^{\ell}.NL_{\diamond}$ is a *residuated category*;
  - (i) the equivalence rel. is a *categorical congruence relation*,
    (ii) the logical introduction rules are *functorial*,
    (iii) the display rules are *adjunctions*.


Step by step

Step 1. Define the smallest categorical equivalence over the *logic of composition* $D^\ell.C$ (i.e. Identity and Cut);

Step 2. Define the category generated by taking formulas as objects and quotienting over *equivalent proofs*.

Theorem

$F(D^\ell.C)$ is a category.

Theorem

Let Ded *be the category of deductive systems and* Cat *be the category of categories. The functor $F\colon \mathrm{Ded} \to \mathrm{Cat}$ is left adjoint to the forgetful functor $U\colon \mathrm{Cat} \to \mathrm{Ded}$, i.e.*

$$\mathrm{Cat}(F(G), \mathcal{C}) \;\cong\; \mathrm{Ded}(G, U(\mathcal{C})).$$

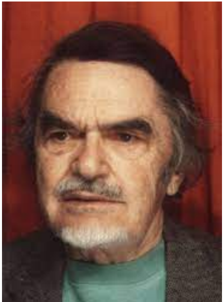
Step 3. Generalize the approach to $F(D^\ell.NL_\diamond)$ and *residuated categories*.

Our results 3/4

- Following Prawitz [10], we defined a **normalization procedure** for $D^{\ell}.NL_{\diamond}$:

two proofs are equivalent
if
they reduce to the same normal form

  - termination and uniqueness of normal forms are guaranteed.
- The induced equivalence relation generates the **Prawitz–Lambek category** P
  - $F \simeq P$.

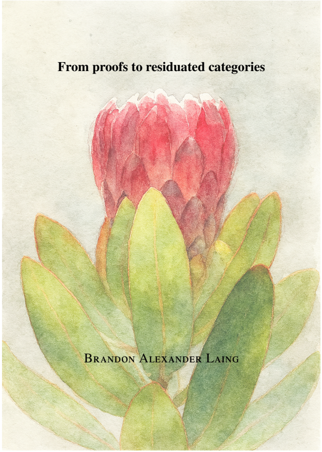


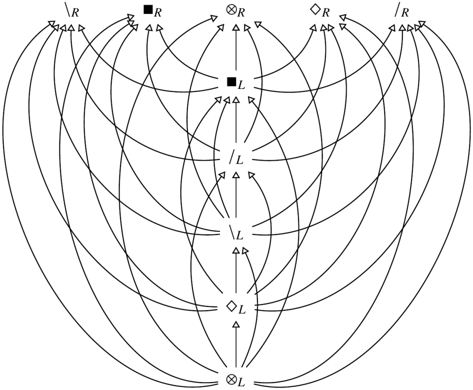
On normalization: permutation “flower”

Step 1. Apply cut-elimination;

Step 2. Permute rules according to a given ordering.

- The def. of *permutable pairs of rules* in $\mathcal{L}(NL_\diamond)$ gives rise to <u>40 cases</u>. In Brandon’s thesis, the following ordering is used:





On normalization: Uniqueness

Lemma (Local Confluence)

$$\begin{tikzcd} & d & \\ b \arrow[ur, dashed, two heads] & & c \arrow[ul, dashed, two heads] \\ & a \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Lemma (Termination)

The reduction system generated by $\rightarrow$ is terminating.

Theorem (Global Confluence)

$$\begin{tikzcd} & d & \\ b \arrow[ur, dashed, two heads] & & c \arrow[ul, dashed, two heads] \\ & a \arrow[ul, two heads] \arrow[ur, two heads] & \end{tikzcd}$$

Proof. By Newman's Lemma using Local confluence and Termination.

Our results 4/4

- We introduced the **D2D algorithm (Display-to-Diagram)**
  - it transforms $D^{\ell}.NL_{\diamond}$-proofs into 1-cell diagrams.

$$\dfrac{C \xrightarrow{h} A \qquad \mu\,\dfrac{\hat{\diamond} A \xrightarrow{g} B}{A \xrightarrow{2} \check{\blacksquare} B}}{C \xrightarrow{1} \check{\blacksquare} B}\circ \quad \equiv \quad \dfrac{\dfrac{\dfrac{C \xrightarrow{h} A}{\hat{\diamond} C \xrightarrow{3} \diamond A}\diamond_R \qquad \diamond_L\dfrac{\hat{\diamond} A \xrightarrow{g} B}{\diamond A \xrightarrow{4} B}}{\hat{\diamond} C \xrightarrow{2} B}\circ}{C \xrightarrow{1} \check{\blacksquare} B}\mu$$

$$\begin{tikzcd}
C \arrow[r, "\eta"] \arrow[d, "h"'] & \blacksquare\diamond C \arrow[r, "\blacksquare\diamond h"] & \blacksquare\diamond A \arrow[dd, "\blacksquare g"] \\
A \arrow[d, "\eta"'] & & \\
\blacksquare\diamond A \arrow[rr, "\blacksquare g"'] & & \blacksquare B
\end{tikzcd}$$

The equational theory shows that the square commutes!

The Lambek program for capturing grammaticality

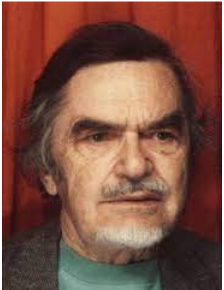
THE MATHEMATICS OF SENTENCE STRUCTURE*

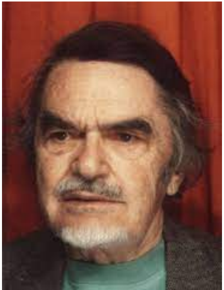
JOACHIM LAMBEK, McGill University

The definitions [of the parts of speech] *are very far from having attained the degree of exactitude found in Euclidean geometry.*

—Otto Jespersen, 1924.

1. Introduction. The aim of this paper is to obtain an effective rule (or algorithm) for distinguishing sentences from nonsentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages such as English, or at least for fragments of such languages. An attempt to formulate such an algorithm is implicit in the work of Ajdukiewicz.† His method, later elaborated by Bar-Hillel [2], depends on a kind of arithmetization of the so-called *parts of speech*, here called *syntactic types*.‡





Formal grammar as a logic

The **basic judgement** of the grammar logic

$$\Gamma \vdash A$$

reads: the structured configuration of linguistic expressions Γ can be categorized as a well-formed expression of type A.

The **basic law of grammatical composition** takes the following form (cfr. Modus Ponens):

- from $\Gamma \vdash A$ and $\Delta \vdash A \backslash B$, infer $\Gamma, \Delta \vdash B$;
- from $\Gamma \vdash B/A$ and $\Delta \vdash A$, infer $\Gamma, \Delta \vdash B$.

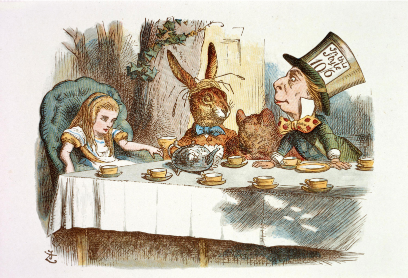
Parsing as deduction

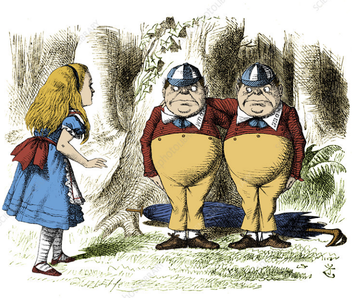
[Lambek 58]: string of words, [Lambek 61]: bracketed strings (phrases)

- Parts of speech (noun, verb...) $\rightsquigarrow$ logical formulas - types.
- Grammaticality judgement $\rightsquigarrow$ logical deduction - computation.

$$\underbrace{np}_{\texttt{time}} \;\hat{\otimes}\; \underbrace{(np\backslash s)}_{\texttt{flies}} \;\hat{\otimes}\; \underbrace{(((np\backslash s)\backslash(np\backslash s))/np}_{\texttt{like}} \;\hat{\otimes}\; \underbrace{(np/n)}_{\texttt{an}} \;\hat{\otimes}\; \underbrace{n}_{\texttt{arrow}} \vdash s$$

Lexicon





- determiner `the`: np/n, adjective `mad`: n/n
- transitive verb `likes`: $(np\backslash s)/np$

  $(\mathtt{The} \cdot (\mathtt{mad} \cdot \mathtt{hatter})) \cdot (\mathtt{likes} \cdot \mathtt{tea})$
- conjunction words `and-but`: *chameleon* word $(X\backslash X)/X$

  $(\mathtt{Tweedledee\ likes\ sweets})_s$ `but` $(\mathtt{Tweedledum\ prefers\ liquor})_s$

  $(\mathtt{likes\ sweets})_{np\backslash s}$ `but` $(\mathtt{hates\ vegetables})_{np\backslash s}$
- relative pronoun `that`: $(n\backslash n)/(s/np)$, i.e. it looks for a noun n to its left and an *incomplete* sentence to its right (s/np: it misses a np, the *gap* at the right)

Deriving a sentence (in Natural Deduction)

$$\dfrac{\dfrac{\texttt{Alice}}{np} \qquad \dfrac{\dfrac{\texttt{found}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\texttt{the}}{np/n} \quad \dfrac{\texttt{key}}{n}}{np}\,/_E}{np\backslash s}\,/_E}{s}\,\backslash_E$$

What about exceptions? ...Controlled structural rules to the rescue!

$D^\ell.NL_\diamond$ can be extended e.g. with the following analytic structural rules α_c (controlled associativity) and κ_c (controlled mixed commutativity):

$$\alpha_c^{-1}\ \frac{X \mathbin{\hat{\otimes}} (Y \mathbin{\hat{\otimes}} \hat{\diamond} Z) \rightarrow W}{(X \mathbin{\hat{\otimes}} Y) \mathbin{\hat{\otimes}} \hat{\diamond} Z \rightarrow W} \quad \alpha_c\ \frac{(\hat{\diamond} X \mathbin{\hat{\otimes}} Y) \mathbin{\hat{\otimes}} Z \rightarrow W}{\hat{\diamond} X \mathbin{\hat{\otimes}} (Y \mathbin{\hat{\otimes}} Z) \rightarrow W} \quad \kappa_c\ \frac{(X \mathbin{\hat{\otimes}} \hat{\diamond} Y) \mathbin{\hat{\otimes}} Z \rightarrow W}{(X \mathbin{\hat{\otimes}} Z) \mathbin{\hat{\otimes}} \hat{\diamond} Y \rightarrow W}$$

These rules have been proposed in linguistic applications to model phenomena of *extraction*.

Let us consider the following lexicon:

- `key` : n, `Alice` : np, `found` : $(np\backslash s)/np$.
- `that`: $(n\backslash n)/(s/\diamond\blacksquare np)$
- `there`: $(np\backslash s)\backslash(np\backslash s)$

$$\dfrac{np \rightarrow np}{\blacksquare np \rightarrow \check{\blacksquare} np}$$

$$\dfrac{np \rightarrow np \qquad s \rightarrow s}{np \backslash s \rightarrow np \check{\backslash} s} \qquad \dfrac{\blacksquare np \rightarrow \check{\blacksquare} np}{\hat{\Diamond} \blacksquare np \rightarrow np}$$

$$(np \backslash s)/np \rightarrow (np \check{\backslash} s)/\hat{\Diamond} \blacksquare np$$

$$(np \backslash s)/np \;\hat{\otimes}\; \hat{\Diamond} \blacksquare np \rightarrow np \check{\backslash} s$$

$$np \;\hat{\otimes}\; ((np \backslash s)/np \;\hat{\otimes}\; \hat{\Diamond} \blacksquare np) \rightarrow s$$

$$\alpha_c^{-1}\;\; (np \;\hat{\otimes}\; (np \backslash s)/np) \;\hat{\otimes}\; \hat{\Diamond} \blacksquare np \rightarrow s$$

$$\hat{\Diamond} \blacksquare np \rightarrow (np \;\hat{\otimes}\; (np \backslash s)/np) \check{\backslash} s$$

$$\Diamond \blacksquare np \rightarrow (np \;\hat{\otimes}\; (np \backslash s)/np) \check{\backslash} s$$

$$(np \;\hat{\otimes}\; (np \backslash s)/np) \;\hat{\otimes}\; \Diamond \blacksquare np \rightarrow s$$

$$np \;\hat{\otimes}\; (np \backslash s)/np \rightarrow s \check{/} \Diamond \blacksquare np$$

$$\dfrac{n \rightarrow n \qquad n \rightarrow n}{n \backslash n \rightarrow n \check{\backslash} n} \qquad np \;\hat{\otimes}\; (np \backslash s)/np \rightarrow s / \Diamond \blacksquare np$$

$$(n \backslash n)/(s/\Diamond \blacksquare np) \rightarrow (n \check{\backslash} n) \check{/} (np \;\hat{\otimes}\; (np \backslash s)/np)$$

$$(n \backslash n)/(s/\Diamond \blacksquare np) \;\hat{\otimes}\; (np \;\hat{\otimes}\; (np \backslash s)/np) \rightarrow n \check{\backslash} n$$

$$\underbrace{n}_{\texttt{key}} \;\hat{\otimes}\; (\underbrace{(n \backslash n)/(s/\Diamond \blacksquare np)}_{\texttt{that}} \;\hat{\otimes}\; (\underbrace{np}_{\texttt{Alice}} \;\hat{\otimes}\; \underbrace{(np \backslash s)/np}_{\texttt{found}})) \rightarrow n$$

$$\dfrac{\dfrac{np \to np \qquad s \to s}{np\backslash s \to np\check{\backslash} s} \qquad \dfrac{\dfrac{np \to np}{\blacksquare np \to \check{\blacksquare} np}}{\hat{\Diamond}\blacksquare np \to np}}{(np\backslash s)/np \to (np\check{\backslash} s)/\hat{\Diamond}\blacksquare np}$$

$$\dfrac{\dfrac{\dfrac{(np\backslash s)/np \to (np\check{\backslash} s)/\hat{\Diamond}\blacksquare np}{(np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np \to np\check{\backslash} s}}{(np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np \to np\backslash s} \qquad \dfrac{np \to np \qquad s \to s}{np\backslash s \to np\check{\backslash} s}}{(np\backslash s)\backslash(np\backslash s) \to ((np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np)\check{\backslash}(np\check{\backslash} s)}$$

$$\dfrac{(np\backslash s)\backslash(np\backslash s) \to ((np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np)\check{\backslash}(np\check{\backslash} s)}{\color{blue}{((np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np) \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s) \to np\check{\backslash} s}}$$

$$\color{blue}{\kappa_c}\ \dfrac{\color{blue}{((np\backslash s)/np \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np) \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s) \to np\check{\backslash} s}}{\color{blue}{((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)) \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np \to np\check{\backslash} s}}$$

$$\dfrac{((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)) \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np \to np\check{\backslash} s}{np \ \hat{\otimes}\ (((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)) \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np) \to s}$$

$$\alpha_c^{-1}\ \dfrac{np \ \hat{\otimes}\ (((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)) \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np) \to s}{(np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s))) \ \hat{\otimes}\ \hat{\Diamond}\blacksquare np \to s}$$

$$\vdots$$

$$\dfrac{\dfrac{n \to n \qquad n \to n}{n\backslash n \to n\backslash n} \qquad \dfrac{\vdots}{np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)) \to s/\Diamond\blacksquare np}}{(n\backslash n)/(s/\Diamond\blacksquare np) \to (n\check{\backslash} n)\check{/}(np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)))}$$

$$\dfrac{(n\backslash n)/(s/\Diamond\blacksquare np) \to (n\check{\backslash} n)\check{/}(np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s)))}{(n\backslash n)/(s/\Diamond\blacksquare np) \ \hat{\otimes}\ (np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s))) \to n\check{\backslash} n}$$

$$\dfrac{(n\backslash n)/(s/\Diamond\blacksquare np) \ \hat{\otimes}\ (np \ \hat{\otimes}\ ((np\backslash s)/np \ \hat{\otimes}\ (np\backslash s)\backslash(np\backslash s))) \to n\check{\backslash} n}{\underbrace{n}_{\texttt{key}} \ \hat{\otimes}\ (\underbrace{(n\backslash n)/(s/\Diamond\blacksquare np)}_{\texttt{that}} \ \hat{\otimes}\ (\underbrace{np}_{\texttt{Alice}} \ \hat{\otimes}\ (\underbrace{(np\backslash s)/np}_{\texttt{found}} \ \hat{\otimes}\ \underbrace{(np\backslash s)\backslash(np\backslash s)}_{\texttt{there}}))) \to n}$$

What about different readings? ...Proofs to the rescue!

$$\frac{\text{Everybody}}{s/(\Diamond\blacksquare np\backslash s)} \otimes \frac{\text{needs}}{((np\backslash s)/((s/np)\backslash s))} \otimes \frac{\text{somebody}}{(s/\Diamond\blacksquare np)\backslash s} \rightarrow s$$

Two proofs use *controlled associativity* ($\Diamond$ and $\blacksquare$ are key to control structural rules) and correspond to the readings [7]:

1. $\forall$-$\exists$ reading: Everybody > somebody > needs
2. $\exists$-$\forall$ reading: Somebody > everybody > needs

Everybody > somebody > needs $\quad (D^{\ell}.NL_{\Diamond} + \alpha_c^{-1} + \alpha_c)$

$$\blacksquare_L\ \dfrac{np_1 \xrightarrow{1_{np}} np_3}{\blacksquare np \to \check{\blacksquare} np} \qquad \mu^{-1}\ \dfrac{\blacksquare np \to \check{\blacksquare} np}{\hat{\Diamond}\blacksquare np \to np}$$

$$\backslash_L\ \dfrac{\hat{\Diamond}\blacksquare np \to np \qquad s_4 \xrightarrow{1_s} s_8}{np\backslash s \to (\hat{\Diamond}\blacksquare np)\check{\backslash} s}$$

$$\blacksquare_L\ \dfrac{np_9 \xrightarrow{1_{np}} np_6}{\blacksquare np \to \check{\blacksquare} np} \qquad \mu^{-1}\ \dfrac{\blacksquare np \to \check{\blacksquare} np}{\hat{\Diamond}\blacksquare np \to np}$$

$$/_L\ \dfrac{s_5 \xrightarrow{1_s} s_7 \qquad \hat{\Diamond}\blacksquare np \to np}{s/np \to s\check{/}(\hat{\Diamond}\blacksquare np)}$$

$$\dfrac{\dfrac{s/np \to s\check{/}(\hat{\Diamond}\blacksquare np)}{\cdots}}{\hat{\Diamond}\blacksquare np \to (s/np)\backslash s}\ \backslash_R$$

$$/_L\ \dfrac{np\backslash s \to (\hat{\Diamond}\blacksquare np)\check{\backslash} s \qquad \hat{\Diamond}\blacksquare np \to (s/np)\backslash s}{(np\backslash s)/((s/np)\backslash s) \to ((\hat{\Diamond}\blacksquare np)\check{\backslash} s)\check{/}(\hat{\Diamond}\blacksquare np)}$$

$$\cdots$$

$$\gamma^{-1}\ \dfrac{}{(\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np)) \to s}$$

$$\alpha_c^{-1}\ \dfrac{(\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np)) \to s}{((\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np) \to s}$$

$$\cdots$$

$$\dfrac{}{(\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)) \to s/(\Diamond\blacksquare np)}\ /_R$$

$$\backslash_L\ \dfrac{(\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)) \to s/(\Diamond\blacksquare np) \qquad s_{10} \xrightarrow{1_s} s_2}{(s_8/(\Diamond\blacksquare np_9))\backslash s_{10} \to (((\hat{\Diamond}\blacksquare np))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\check{\backslash} s}$$

$$\gamma^{-1}\ \dfrac{(s_8/(\Diamond\blacksquare np_9))\backslash s_{10} \to (((\hat{\Diamond}\blacksquare np))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\check{\backslash} s}{(((\hat{\Diamond}\blacksquare np))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\,\hat{\otimes}\,(s/(\Diamond\blacksquare np))\backslash s \to s}$$

$$\alpha_c\ \dfrac{(((\hat{\Diamond}\blacksquare np))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\,\hat{\otimes}\,(s/(\Diamond\blacksquare np))\backslash s \to s}{(\hat{\Diamond}\blacksquare np)\,\hat{\otimes}\,(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,((s/(\Diamond\blacksquare np))\backslash s)) \to s}$$

$$\cdots$$

$$\dfrac{}{((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,((s/(\Diamond\blacksquare np))\backslash s) \to (\Diamond\blacksquare np)\backslash s}\ \backslash_R$$

$$/_L\ \dfrac{s_0 \xrightarrow{1_s} s_{11} \qquad ((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,((s/(\Diamond\blacksquare np))\backslash s) \to (\Diamond\blacksquare np)\backslash s}{s_0/((\Diamond\blacksquare np_1)\backslash s_2) \to s\check{/}(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,((s/(\Diamond\blacksquare np))\backslash s))}$$

$$\beta^{-1}\ \dfrac{s_0/((\Diamond\blacksquare np_1)\backslash s_2) \to s\check{/}(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,((s/(\Diamond\blacksquare np))\backslash s))}{\underbrace{s_0/((\Diamond\blacksquare np_1)\backslash s_2)}_{\text{Everybody}}\ \hat{\otimes}\ \underbrace{(((np_3\backslash s_4)/((s_5/np_6)\backslash s_7))}_{\text{needs}}\ \hat{\otimes}\ \underbrace{((s_8/(\Diamond\blacksquare np_9))\backslash s_{10}))}_{\text{somebody}} \to s_{11}}$$

Somebody > everybody > needs $(D^{\ell}.NL_{\Diamond} + \alpha_c^{-1})$

$$
\begin{prooftree}
\AxiomC{$s_0 \xrightarrow{1_s} s_8$}
\AxiomC{$np_1 \xrightarrow{1_{np}} np_3$}
\LeftLabel{$\blacksquare_L$}
\UnaryInfC{$\blacksquare np \rightarrow \check{\blacksquare} np$}
\noLine
\UnaryInfC{$\cdots$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$\Diamond\blacksquare np \rightarrow np$}
\AxiomC{$s_4 \xrightarrow{1_s} s_2$}
\LeftLabel{$\backslash_L$}
\BinaryInfC{$np\backslash s \rightarrow (\Diamond\blacksquare np)\check{\backslash} s$}
\RightLabel{$\backslash_R$}
\UnaryInfC{$np\backslash s \rightarrow (\Diamond\blacksquare np)\backslash s$}
\AxiomC{$s_5 \xrightarrow{1_s} s_7$}
\AxiomC{$np_9 \xrightarrow{1_{np}} np_6$}
\LeftLabel{$\blacksquare_L$}
\UnaryInfC{$\blacksquare np \rightarrow \check{\blacksquare} np$}
\UnaryInfC{$\hat{\Diamond}\blacksquare np \rightarrow np$}
\LeftLabel{$/_L$}
\BinaryInfC{$s/np \rightarrow s\check{/}(\hat{\Diamond}\blacksquare np)$}
\noLine
\UnaryInfC{$\cdots$}
\RightLabel{$\backslash_R$}
\UnaryInfC{$\hat{\Diamond}\blacksquare np \rightarrow (s/np)\backslash s$}
\LeftLabel{$/_L$}
\BinaryInfC{$(np\backslash s)/((s/np)\backslash s) \rightarrow ((\Diamond\blacksquare np)\backslash s)\check{/}(\hat{\Diamond}\blacksquare np)$}
\noLine
\UnaryInfC{$\cdots$}
\RightLabel{$\backslash_R$}
\UnaryInfC{$((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np) \rightarrow (\Diamond\blacksquare np)\backslash s$}
\LeftLabel{$/_L$}
\BinaryInfC{${\color{blue}s_0/((\Diamond\blacksquare np_1)\backslash s_2)} \rightarrow s\check{/}(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np))$}
\LeftLabel{β^{-1}}
\UnaryInfC{$(s/((\Diamond\blacksquare np)\backslash s))\,\hat{\otimes}\,(((np\backslash s)/((s/np)\backslash s))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np)) \rightarrow s$}
\LeftLabel{${\color{blue}\alpha_c^{-1}}$}
\UnaryInfC{$((s/((\Diamond\blacksquare np)\backslash s))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\,\hat{\otimes}\,(\hat{\Diamond}\blacksquare np) \rightarrow s$}
\noLine
\UnaryInfC{$\cdots$}
\RightLabel{$/_R$}
\UnaryInfC{$(s/((\Diamond\blacksquare np)\backslash s))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)) \rightarrow s/(\Diamond\blacksquare np)$}
\AxiomC{$s_{10} \xrightarrow{1_s} s_{11}$}
\LeftLabel{$\backslash_L$}
\BinaryInfC{${\color{red}(s_8/(\Diamond\blacksquare np_9))\backslash s_{10}} \rightarrow ((s/((\Diamond\blacksquare np)\backslash s))\,\hat{\otimes}\,((np\backslash s)/((s/np)\backslash s)))\check{\backslash} s$}
\LeftLabel{γ^{-1}}
\UnaryInfC{$\underbrace{s_0/((\Diamond\blacksquare np_1)\backslash s_2)}_{\color{blue}\textit{Everybody}}\ \hat{\otimes}\ \underbrace{(((np_3\backslash s_4)/((s_5/np_6)\backslash s_7))}_{\textit{needs}}\ \hat{\otimes}\ \underbrace{((s_8/(\Diamond\blacksquare np_9))\backslash s_{10}))}_{\color{red}\textit{somebody}} \rightarrow s_{11}$}
\end{prooftree}
$$

Conclusions

We have developed a *principled* (i.e. uniform and modular) framework (proof theory + categorical semantics) [8].

We plan to generalize the approach to

- any **basic displayable logic** and capture *analytic-inductive* [5] or *inductive* [3] **axiomatic extensions**;
- normalization procedures leading to **pre-normal form proofs** [2] or **focalized proofs** [6];
- multi-type proof theory [4], where **sequents are parametrized with agents** and properly interact with modalities;
- **weighted proof theory** and (quantale) **enriched categories** [cfr. Apostolos' previous talk];
- **refutational calculi** [cfr. Andrea's next talk].

References I

[1] Nuel D. Belnap Jr.
Display logic.
Journal of Philosophical Logic, 11:375–417, 1982.

[2] Jinsheng Chen, Giuseppe Greco, Alessandra Palmigiano, and Apostolos Tzimoulis.
Syntactic completeness of proper display calculi.
ACM Transactions on Computational Logic, 23(4):1–46, 2022.

[3] Andrea De Domenico, Giuseppe Greco, and Alessandra Palmigiano.
Inception display calculi.
Manuscript submitted for publication.

References II

[4] Sabine Frittella, Giuseppe Greco, Alexander Kurz, Alessandra Palmigiano, and Vlasta Sikimić.
Multi-type sequent calculi.
In A. Indrzejczak, J. Kaczmarek, and M. Zawidski, editors, *Proceedings Trends in Logic XIII*, volume 13, pages 81–93, 2014.

[5] Giuseppe Greco, Minghui Ma, Alessandra Palmigiano, Apostolos Tzimoulis, and Zhiguang Zhao.
Unified correspondence as a proof-theoretic tool.
Journal of Logic and Computation, 28(7):1367–1442, 2016.

[6] Giuseppe Greco, Michael Moortgat, Valentin D. Richard, and Apostolos Tzimoulis.
Lambek–Grishin calculus: Focusing, display and full polarization.
In *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, pages 877–915. Springer, 2023.

References III

[7] Natasha Kurtonina and Michael Moortgat.
Structural control.
Specifying syntactic structures, pages 75–113, 1997.

[8] Brandon Laing.
From Proofs to Residuated Categories.
PhD thesis, Stellenbosch University and Vrije Universiteit Amsterdam, 2025.
PhD thesis.

[9] Joachim Lambek.
Deductive systems and categories: I. Syntactic calculus and residuated categories.
Mathematical systems theory, 2(4):287–318, 1968.

[10] Dag Prawitz.
Natural deduction: A proof-theoretical study.
Courier Dover Publications, 2006.