Soberness as idempotent-completeness

towards a formal model theory of virtual ultracategories

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VU-CATEGORIES

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We propose to analyse vu-categorically the *topological* notion of soberness, and to see how it relates to the *logical* notion of axiomatizable class, and to the *categorical* notion of retraction.

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Ultrafiltered-shaped arrows

The core idea of vu-categories is to consider ultrafilter-shaped arrows:

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Theses are called vu-arrows, or σ-arrows to specify the ultrafilter.

They can be thought either as

- witnesses that the points (q_s) converges to the point p along σ.
- homomorphisms of models from p to the ultraproduct of the (q_s) along σ.

Vu-categories: definition

A vu-category $\mathcal{X}$ consists in:

- a class of points,
- for each point p, ultrafilter σ on S, and S-family of points (q_s), a set $\mathcal{X}_\sigma(p, q_s)$ of vu-arrows,

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- identities and composites,

$$p \xrightarrow{\mathrm{i}_p}\!\!\!\prec_{\star} p \qquad p \xrightarrow{u}\!\!\!\prec_{\sigma} \left(q_s \xrightarrow{v_s}\!\!\!\prec_{\tau_s} r_{s,t}\right) = p \xrightarrow{v_s \circ_\sigma u}\!\!\!\prec_{\sigma \cdot \tau_s} r_{s,t}$$

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- for each morphism of ultrafilters $\phi : \tau \twoheadrightarrow \sigma$, a thickening operation,

$$\begin{aligned} \mathcal{X}_\sigma(p, q_s) &\longrightarrow \mathcal{X}_\tau(p, q_{\phi t}) \\ u &\longmapsto \phi^* u \end{aligned}$$

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- equations between these operations: neutrality of the identities, associativity of the composition, functoriality of thickening, and naturality of the composition in the thickening.

Vu-categories: examples

A topological space T has a posetal vu-structure, $T_\sigma(p, q_s) = [\![p \preccurlyeq_\sigma q_s]\!]$.

where we write $p \preccurlyeq_\sigma q_s$ if the points $(q_s)_{s \in S}$ converge to p along σ.

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A 1-category $\mathcal{C}$ can be see as a vu-category[1], $\mathcal{C}_\sigma(p, q_s) = \int_{s:\sigma} \mathcal{C}(p, q_s)$

where $\int_{s:\sigma}$ denotes the ultraproduct of sets along σ.

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In particular, we get a vu-structure on Set seen as an ultracategory.

And for a topological space T we have $\mathrm{Sh}(T) \simeq \mathrm{vuCat}(T, \mathrm{Set})$.

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Reconstruction theorem

For $\mathcal{E}$ a topos[1], we have a vu-structure on the points of $\mathcal{E}$ with vu-arrows $p \multimap_\sigma q_s$ are given by natural transformation $p^* \Rightarrow \int_{s:\sigma} {q_s}^*$.

There is an adjunction[2] induced by the dualizing object Set,

$$\mathrm{pt} : \mathrm{Topos} \leftrightarrows \mathrm{vuCat_b} : \mathrm{Sh}$$

[1] our toposes will always be Grothendieck toposes

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Reconstruction theorem [Saa25; Ham25; GMT26]

The counit $\mathrm{Sh}(\mathrm{pt}(\mathcal{E})) \hookrightarrow \mathcal{E}$ is the restriction of a topos $\mathcal{E}$ to all its points.

A topos with enough points $\mathcal{E}$ can be reconstructed from its vu-category of points, $\mathcal{E} \simeq \mathrm{Sh}(\mathrm{pt}(\mathcal{E}))$.

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There are some interesting examples of vu-categories that are not points of a topos, for example the ultracategory of complete metric spaces.

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SUBCLOSURE

Subclosure

Recall that a topos $\mathcal{E}$ can be seen as classifying a geometric theory whose models are given by the points of the topos $\mathrm{pt}(\mathcal{E}) = \mathrm{Topos}(\mathrm{Set}, \mathcal{E})$.

Let $\mathbb{A} \subseteq \mathrm{pt}(\mathcal{E})$ be a class of points of a topos $\mathcal{E}$.

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Let $\mathbb{A} \subseteq \mathrm{pt}(\mathcal{E})$ be a class of points of a topos $\mathcal{E}$.

We define the subclosure of $\mathbb{A}$, $\mathbb{A} \subseteq \mathrm{scl}(\mathbb{A}) \subseteq \mathrm{pt}(\mathcal{E})$,

- $\mathrm{scl}(\mathbb{A})$ consists of the points of $\mathcal{E}$ that belongs to all subtoposes of $\mathcal{E}$ containing $\mathbb{A}$,

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- or equivalently, $\mathrm{scl}(\mathbb{A}) = \mathrm{pt}(\mathcal{E}_{\restriction A})$ with $\mathcal{E}_{\restriction A} \hookrightarrow \mathcal{E}$ is the subtopos of $\mathcal{E}$ induced[1] by $\mathbb{A}$,

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  if $\mathbb{A}$ is small, $\mathcal{E}_{\restriction A}$ is given by the factorisation $\mathrm{Set}^{\mathbb{A}} \twoheadrightarrow \mathcal{E}_{\restriction A} \hookrightarrow \mathcal{E}$.

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From a logical point of view, $\mathbb{A}$ is a collection of models of the theory classified by $\mathcal{E}$, and $\mathrm{scl}(\mathbb{A})$ is the class of models satisfying the geometric theory[1] of $\mathbb{A}$,

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$\mathsf{scl}(\mathbb{A}) = \mathbb{A}$ means that $\mathbb{A}$ is geometrically axiomatizable,

and if $\mathcal{E}$ has enough points, $\mathsf{scl}(\mathbb{A}) = \mathsf{pt}(\mathcal{E})$ iff $\mathbb{A}$ is separating[2].

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Questions

Which models satisfy the theory of $\mathbb{A}$? i.e. can we describe $\mathrm{scl}(\mathbb{A})$?

When is $\mathbb{A}$ geometrically axiomatizable? i.e. $\mathrm{scl}(\mathbb{A}) \overset{?}{=} \mathbb{A}$

When is $\mathbb{A}$ separating? i.e. $\mathrm{scl}(\mathbb{A}) \overset{?}{=} \mathrm{pt}(\mathcal{E})$

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The 0-dimensional case: topological spaces

Let T be a topological space and $A \subseteq T$ a subset of points,

$$A \subseteq \mathrm{scl}_T(A) \subseteq T,$$

if T is sober, $\mathrm{scl}(A)$ is the smallest sober subspace of T containing A.

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Proposition[1] **[Joh80]**

A point p is in $\mathrm{scl}(A)$ iff every locally closed set[2] containing p intersects A.

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$$p \in \mathsf{scl}(A) \iff p \in \mathsf{cl}(A \cap \mathsf{cl}\{p\}).$$

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Proposition

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The Alexandroff case: presheaf toposes

We now focus on the case $\mathcal{E} = \mathsf{Set}^{\mathcal{C}}$ for $\mathcal{C}$ a small category,

$$\mathcal{C} \subseteq \mathsf{Ind}(\mathcal{C}) = \mathsf{pt}(\mathcal{E}).$$

Proposition [Joh02, A4.2.7(b)]

For $\mathbb{A} \subseteq \mathcal{C} \subseteq \mathsf{pt}(\mathcal{E})$, $\mathbb{A}$ is separating in $\mathcal{E}$ if and only if each $p \in \mathcal{C}$ is a retract of some $a \in \mathbb{A}$.

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Remark

For an arbitrary class $\mathbb{A} \subseteq \mathsf{pt}(\mathcal{E})$, "$\mathbb{A}$ is filtered-colimit-dense"[1] is a sufficient condition for $\mathbb{A}$ to be separating in $\mathcal{E}$, but it is hard to find a condition that is also necessary.

[1] any point of $\mathcal{E}$ is a filtered colimit of points in $\mathbb{A}$

Vu-retractions

Hence retractions play a particular role.

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Definition: vu-retraction

In a vu-category, a point p is a vu-retract of a σ-family of points $(a_s)_{s:\sigma}$ if there are $i : p \longrightarrow\!\!\prec_\sigma a_s$ and $(r_s : a_s \longrightarrow\!\!\prec_{\tau_s} p)$ such that $r_s \circ_\sigma i : p \longrightarrow\!\!\prec_{\sigma \cdot \tau_s} p$ is the diagonal[1].

The pair (i, r_s) is called a vu-retraction.

[1] that is the thickening of the identity at p

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- In $A \subseteq T$ a topological space, then p is a vu-retract of point $(a_s)_{s:\sigma}$ if and only if $p \preccurlyeq_\sigma a_s$ and $\forall_s (a_s \preccurlyeq p)$.

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- In a 1-category $\mathcal{C}$, if p is a retract of a in the 1-categorical sense, then p is a vu-retract of the constant vu-family $(a)_{\star}$.

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Definition: vu-retraction

A point p is a vu-retract of a σ-family of points $(a_s)_{s:\sigma}$ if there are $i : p \multimap_\sigma a_s$ and $(r_s : a_s \multimap_{\tau_s} p)$ such that $r_s \circ_\sigma i$ is the diagonal.

- In $A \subseteq T$ a topological space, then p is a vu-retract of point $(a_s)_{s:\sigma}$ if and only if $p \preccurlyeq_\sigma a_s$ and $\forall_s (a_s \preccurlyeq p)$.
- In a 1-category $\mathcal{C}$, if p is a retract of a in the 1-categorical sense, then p is a vu-retract of the constant vu-family $(a)_\star$.
- In an ultracategory, A is an ultraretract of its ultrapower $(A^\sigma)_\star$,

  $i : A \multimap_\star A^\sigma$ is given by the diagonal $A \longrightarrow A^\sigma$,
  $r : A^\sigma \multimap_\sigma A$ is given by the identity $A^\sigma \longrightarrow A^\sigma$.

Filtered colimits

Vu-retractions have enough expressive power to see filtered colimits,

categorically, retractions are filtered colimits,
vu-categorically, filtered vu-colimits[1] are vu-retractions.

[1] i.e., the universal property holds for vu-arrows as well

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Claim

If p is the vu-colimit of a filtered diagram $(a_i)_{i:I}$, then p is a vu-retract of $(a_i)_{i:\sigma}$ for any final ultrafilter σ on I.

The $(r_i : a_i \prec_\star p)_i$ are given by the legs of the universal cocone,
$i : p \prec_{i:\sigma} a_i$ arises from the σ-cocone $(a_{i_0} \prec_{i:\sigma} a_i)_{i_0:I}$ given by the composites

$$a_{i_0} \prec_{i:\sigma} (a_{i_0} \xrightarrow{a_{i_0 \leq i}}\!\!\prec_\star a_i).$$

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This explains why vu-functors automatically preserves filtered colimits:
filtered colimits are absolute in the vu-sense[2].

[1] i.e., the universal property holds for vu-arrows as well
[2] assuming tautness

The main theorem

For topological spaces, $p \in \mathrm{scl}(A)$ if and only if p is a vu-retract of A.

We categorify this result.

Theorem (Saadia–Yuksel)

For a class $\mathbb{A} \subseteq \mathrm{pt}(\mathcal{E})$, $p \in \mathrm{scl}(\mathbb{A})$ iff p is a vu-retract of points in $\mathbb{A}$.

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We can now answer the questions from earlier,

$\mathbb{A}$ geometrically axiomatizable iff $\mathbb{A}$ is closed under vu-retractions in $\mathrm{pt}(\mathcal{E})$,

for $\mathcal{E}$ with enough point, $\mathbb{A}$ is separating in $\mathcal{E}$ iff every points of $\mathcal{E}$ is a vu-retract of points in $\mathbb{A}$.

Vu-subdense

In particular, we get a candidate for vu-functors between vu-categories that correspond to geometric surjections.

A vu-functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is vu-subdense if every point of $\mathcal{Y}$ is a vu-retract of points in the image of F.

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Theorem (Saadia–Yuksel)

- Between toposes with enough points, a geometric morphism f is surjective if and only if $\mathrm{pt}(f)$ is vu-subdense.
- The unit $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is always vu-subdense.

OTHER CLASSES OF GEOMETRIC MORPHISMS

Subhyperconnectedness

Vu-subdense vu-functors are the vu-analogues of geometric surjections.

We now study the vu-analogues of hyperconnected geometric morphisms.

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It is convenient to consider a slightly more general class.

A geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is subhyperconnected if it is a composite of morphisms which are either hyperconnected or embeddings.

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Proposition [Joh02, A4.6.10]

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is full iff $\mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{Set}^{\mathcal{D}}$ is subhyperconnected.

The vu-functors corresponding to subhyperconnected morphisms should encode a notion of fullness.

Vu-fullness

A vu-functor is said vu-full if it is full up to thickening.

Definition: vu-fullness

A vu-functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is vu-full if each $v : F(p) \relbar\!\!\prec_{\sigma} F(q_s)$ in $\mathcal{Y}$ admits a $u : p \relbar\!\!\prec_{\tau} q_{\phi t}$ in $\mathcal{X}$ such that $F(u) = \phi^* v$, with $\phi : \tau \twoheadrightarrow \sigma$ in $\mathbb{U}$.

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By a similar proof argument than the theorem relating geometric surjections and vu-subdense vu-functors, we can prove.

Theorem (Saadia–Yuksel)

- Between toposes with enough points, a geometric morphism f is subhyperconnected if and only if $\mathrm{pt}(f)$ is vu-full.
- The unit $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is always vu-full.

Embedding & localic geometric morphisms

We have also vu-analogues for embeddings and for localic morphisms.

A vu-functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is vu-fully-faithful (resp. vu-faithful) if $\mathcal{X}_\sigma(p, q_s) \longrightarrow \mathcal{Y}_\sigma(p, q_s)$ are bijections (resp. injections).

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Between toposes with enough points, a geometric morphism f is an embedding (resp. is localic) if and only if $\mathrm{pt}(f)$ is vu-fully-faithful (resp. vu-faithful).

The two direct implications follow from an easy computation.

The converse for embeddings follows from the reconstruction theorem.

The converse for localic maps follows from the above result for hyperconnected by using the hyperconnected–localic factorization.

Enough sheaves

We say that $\mathcal{X}$ has enough sheaves if $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is vu-faithful, i.e., if two parallel vu-arrows of $\mathcal{X}$ inducing the same action on sheaves are equals.

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Knowing that $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ always vu-full, we can deduce:

For a vu-category[1] $\mathcal{X}$, the following are equivalent:

- $\mathcal{X}$ has enough sheaves.
- $\mathcal{X} \hookrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is vu-fully-faithful.
- $\mathcal{X} \simeq \mathrm{pt}(\mathcal{E})_{\restriction \mathbb{A}}$ for some $\mathbb{A} \subseteq \mathrm{pt}(\mathcal{E})$.
- $\mathcal{X}$ admits a vu-faithful functor $\mathcal{X} \longrightarrow \mathrm{Set}$ into the vu-category of sets.

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There are some vu-categories that does not have enough sheaves, for example the ultracategory of complete metric spaces[2].

---

[1] $\mathcal{X}$ is assumed bounded and taut

[2] see Henry's slides [Hen26]

Summary and questions

For f is a geometric morphism between toposes with enough points,

f is	localic	subhyperconn.	surjective	embedding
iff $\mathrm{pt}(f)$ is	vu-faithful	vu-full	vu-subdense	vu-fully-faithful

Moreover the unit $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is always vu-full and vu-subdense,

that is $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is surjective on the objects *up to vu-retractions*, and full on the vu-arrows *up to thickening*.

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that is $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ is surjective on the objects *up to vu-retractions*, and full on the vu-arrows *up to thickening*.

- On the objects $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ should 'splits the vu-idempotents'. Is there a nice notion of vu-idempotents? of vu-Cauchy completion?
- How does $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ identify two vu-arrows? When is $\mathcal{X} \longrightarrow \mathrm{pt}(\mathrm{Sh}(\mathcal{X}))$ also vu-faithful?
- What about other classes of geometric morphisms? Open, proper, connected, locally connected . . .

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