A topos for étale-finite Heyting algebras

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Lattices of truth-values in toposes

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3. The effective topos Eff.

Lattices of truth-values

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General problem: What are the possible lattices which arise in this way?

All lattices of truth-values are *Heyting algebras*:

Definition

An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is called a Heyting algebra if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

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Complete Heyting algebras are called *frames* or *locales*: for example

1. $\mathcal{O}(X)$ – opens of a topological space X;
2. $\mathrm{Con}(A)$ – congruence lattices for congruence-distributive algebras.

Which Heyting algebras

For L a locale, we have:

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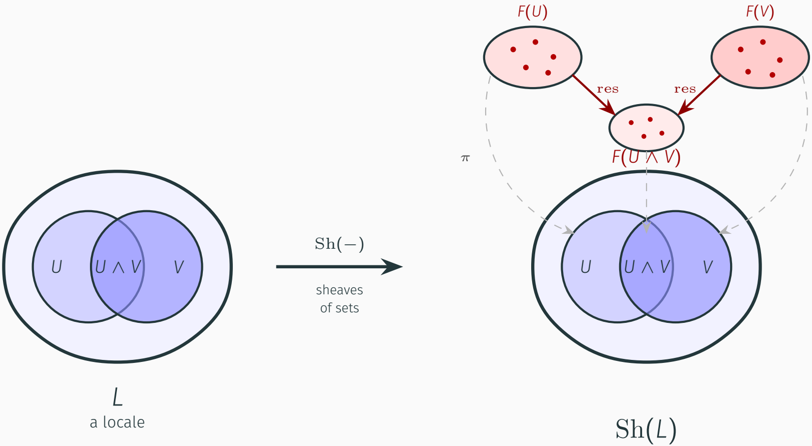

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Logically: interpreting a higher order theory in its propositional fragment – analogous to **higher-order uniform interpolation**.

Topos-theoretically: is there a **good quotient of free toposes** which produces the free Heyting algebras $\mathcal{F}(X)$?

We don’t know which Heyting algebras

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This was the approach suggested by D. Pataraia. But eliminating higher-order sentences is much more difficult: equality types, extensionality, and various other headaches.

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(Please let us know if you do!)

Étale-finite Heyting algebras

We work with a class introduced by E. Kuznetsov (2024):

Definition (Étale-finite Heyting algebra)

A Heyting algebra H is étale-finite if there is a finite Heyting algebra H_0 and a Heyting algebra homomorphism $f\colon H_0 \to H$ satisfying Jibladze's law: for every $x \in H$:

$$\bigvee_{x_0 \in H_0} \left(x \leftrightarrow f(x_0)\right) = 1.$$

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Conceptually: there are finitely many definable truth-values.

Étale-finite Heyting algebras and Spaces

Example

Finite algebras are étale-finite. For subdirectly irreducible, étale-finiteness is just finiteness.

We also work with their duals:

Definition (Esakia space)

An ordered space $(X, \leq, \tau)$ is an Esakia space if it is (1) compact, (2) Priestley separation and (3) $\downarrow W$ is clopen whenever W is clopen.

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$$f(x) \leq y \implies \exists! x' \geq x, f(x') = y;$$

equivalently $\uparrow x \cong_f \uparrow f(x)$.

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We work with maps which are local homeomorphisms both for the order and the topology:

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Proposition (Abbadini, A., Arrieta)

f is a spectral local homeomorphism if and only if it is a local homeomorphism for the Priestley/patch topology, and a strict p-morphism.

Building toposes

Our analogue of $\mathsf{Top}_{\mathrm{LH}}$ – topological spaces with local homeomorphisms – is then

$$\mathsf{Esa}_{\mathrm{SLH}}$$

Category of Objects

Our analogue of $\mathsf{Top}_{\mathrm{LH}}$ – topological spaces with local homeomorphisms – is then

$$\mathsf{Esa}_{\mathrm{SLH}}$$

Not such a nice category: lacks terminal objects and equalizers. But its slices are good.

For X an Esakia space:

$$\mathsf{Esa\acute{E}t}(X) = \mathsf{Esa}_{\mathrm{SLH}}/X$$

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For every Esakia space X, the category $\mathsf{EsaÉt}(X)$ *is finitely complete:*

1. *(Terminal object) The identity* $1_X \colon X \to X$ *is the terminal object in* $\mathsf{EsaÉt}(X)$.

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2. *(Binary product) If $f \colon Y \to X$ and $g \colon Z \to X$ are two objects in $\mathsf{Esa\acute{E}t}(X)$, then the pullback $Y \times_X Z$ as ordered spaces with the two projection maps is the product in $\mathsf{Esa\acute{E}t}(X)$.*

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3. *(Equaliser) Whenever $f \colon Y \to X$ and $g \colon Z \to X$ are two objects of $\mathsf{Esa\acute{E}t}(X)$, and $p_1, p_2 \colon Y \to Z$ are two spectral local homeomorphisms making the two triangles in the diagram commute,*

$$\begin{tikzcd} Y \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

then

$$\mathrm{Eq}(p_1, p_2) = \{y \in Y : p_1(y) = p_2(y)\},$$

together with the inclusion into Y, is the equaliser of p_1 and p_2 in $\mathsf{Esa\acute{E}t}(X)$.

Palate cleanser - subobject lattice

Any inclusion of a clopen upset $U \hookrightarrow X$ is a spectral local homeomorphism, and these are all the injective ones. So we have:

Theorem (Abbadini, A., Arrieta)

Given an Esakia space, $\mathcal{E} = \mathbf{EsaÉt}(X)$*, we have:*

$$\mathrm{Sub}_{\mathcal{E}}(1) \cong \mathrm{ClopUp}(X).$$

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Via Esakia duality, this gives us every Heyting algebra. Incidentally, $\mathbf{Esa\acute{E}t}(X)$ is always a Heyting pretopos.

What about power objects?

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More difficult to construct. Basic idea: use a Vietoris-like construction (typical in the theory of (co)free objects in theories such as Heyting algebras and modal algebras).

Power objects – definition

Given $f\colon Y \to X$ an object in $\mathbf{Esa\acute{E}t}(X)$, where X is étale-finite:

$$P_f(Y) := \{(x, K) : x \in X,\ K \text{ an upset of } Y \text{ with } K \subseteq f^{-1}[{\uparrow}x]\}.$$

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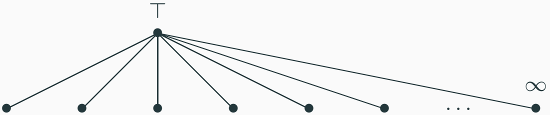
we define the partial order $\preceq$ by setting

$$(x, K) \preceq (x', K') \iff x \leq x' \text{ and } K' = K \cap f^{-1}[\uparrow x'].$$

and let it inherit the topology from the product $X \times \mathcal{V}(Y)$ – the Vietoris space of Y.

Power objects – Example

Take the space:



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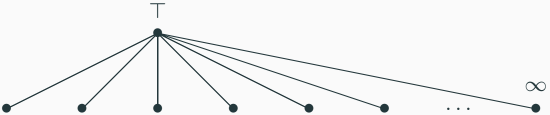
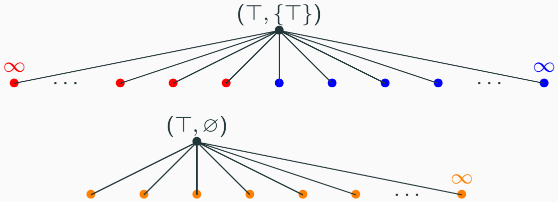


Then $P_{1_X}(X)$ will be:



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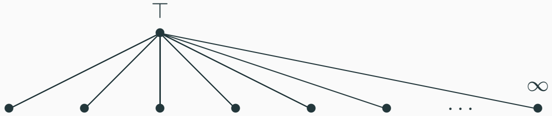

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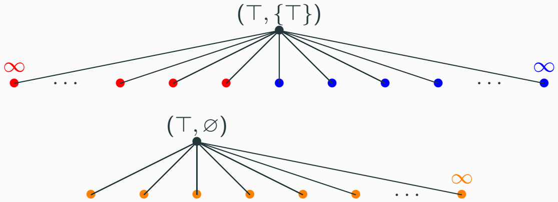

Blue points – $(x, \{\top\})$;

Red points – $(x, \{x, \top\})$.

Orange points – $(x, \varnothing)$.

Power objects are appropriate

Theorem (Abbadini, A., Arrieta)

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The key difficulty lies in showing that $P_f(Y)$ is actually a compact space. But with that we obtain:

Theorem (Elementary topos for étale-finite)

For every étale-finite Heyting algebra H, with $X = \mathrm{Spec}(H)$, $\mathcal{E} = \mathbf{Esa\acute{E}t}(X)$ is an elementary topos, such that $\mathrm{Sub}_{\mathcal{E}}(1) \cong H$.

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Theorem (Abbadini, A., Arrieta)

For every Esakia space X, the category $\mathbf{EsaEt}(X)$ *is finitely propositional.*

Finitely propositional are all we can get this way

Through studying the structure of the lattice of truth-values, we get (Abbadini, A., Almeida):

Theorem (Finitely propositional $\leftrightarrow$ étale-finite)

For a Heyting algebra H, the following are equivalent:

1. $\mathbf{Esa\acute{E}t}(\mathrm{Spec}(H))$ *is a topos;*
2. *H is isomorphic to the lattice of truth values of some finitely propositional topos;*
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This hints at a possible (2-)equivalence of categories, which we have not yet investigated.

Final remarks

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If there is a counterexample, it should morally be the free Heyting algebra $\mathcal{F}(\omega)$, so it should fight us on one of these fronts.

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1. Arithmetic and type theory: What happens to these questions if we demand the presence of a natural numbers objects?
2. Higher topos theory: What happens if one replaces lattices of truth-values by a "large" object, like an elementary topos in a 2-topos?

Thank you!

Questions?