The equivalential fragment of $\mathbf{FL}_{ew}$

Gaia Petreni

University of Siena

Joint work with Tomasz Kowalski and Sara Ugolini

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Introduction

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Examples:

- Classical logic
- Intuitionistic logic
- Many-valued logics

Their analysis under algebraic perspective allowed to recognize an important common feature of these logics: the *residuation property*. The corresponding algebraic models are *residuated structures*.

FL-algebras

Nowadays, substructural logics are seen as the extensions of the *Full Lambek Calculus* **FL**.

Equivalent algebraic semantics: variety of **FL**-*algebras*, i.e. expansions $\{\cdot, \backslash, /, \wedge, \vee, 0, 1\}$ of residuated lattices with an additional constant operation 0.

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Example

Classical, intuitionistic, and many-valued logics, fit into the framework of substructural logics with exchange and weakening ($\mathbf{FL_{ew}}$), whose algebraic models are commutative integral (1 is the greatest element) and 0-bounded (0 is the least element) FL-algebras.

Equivalence

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Consider the equivalence connective $\leftrightarrow$ defined by

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

We study the subset of formulas of $\mathbf{FL_{ew}}$ that can be rewritten using only $\leftrightarrow$ and we restrict the semantic to this connective only.

Remark

$\leftrightarrow$ allows us to express **equality at the propositional level**: for every x, y in any $\mathbf{FL}_{ew}$-algebra,

$$x = y \quad \text{if and only if} \quad x \leftrightarrow y = 1.$$

Existing work

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- Wroński and Kabziński (1975): study of the equivalential fragment of intuitionistic logic from an algebraic point of view.

  They introduce the notion of *equivalential algebra*, proving that the algebraic $\leftrightarrow$-subreducts of Heyting algebras are a variety axiomatized by equations which are an algebraic translation of Tax's axioms.

Our contributions

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In particular,

- we provide a sequent calculus for $\mathbf{FL}_{\mathbf{ew}}^{\leftrightarrow}$;
- establish that it satisfies cut elimination: provability is decidable in $\mathbf{FL}_{\mathbf{ew}}^{\leftrightarrow}$;
- describe some properties.

A *sequent* is a pair (Γ, φ) where Γ is a finite multiset of formulas, and φ is a formula.

We will now define two sequent calculi. The first, called $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$, is the $\{\wedge, \to\}$-fragment of the usual sequent calculus for $\mathbf{FL_{ew}}$.

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cut:

$$\frac{\Gamma \Rightarrow \alpha \qquad \Sigma, \alpha \Rightarrow \varphi}{\Gamma, \Sigma \Rightarrow \varphi}\ (\text{cut})$$

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weakening:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi} \text{ (i)}$$

and rules for connectives:

$$\frac{\Gamma, \alpha \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\,(\wedge\Rightarrow)_1 \qquad \frac{\Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\,(\wedge\Rightarrow)_2$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\,(\Rightarrow\wedge)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\,(\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Delta, \beta \Rightarrow \varphi}{\Gamma, \Delta, \alpha \to \beta \Rightarrow \varphi}\,(\to\Rightarrow)$$

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- the same initial sequent schema of $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$
- cut
- weakening
- the rules for equivalence

$$\frac{\Gamma, \alpha \Rightarrow \beta \qquad \Gamma, \beta \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \leftrightarrow \beta}\ (\Rightarrow\leftrightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Delta, \beta \Rightarrow \varphi}{\Gamma, \Delta, \alpha \leftrightarrow \beta \Rightarrow \varphi}\ (\leftrightarrow\Rightarrow)_1 \qquad \frac{\Gamma \Rightarrow \beta \qquad \Delta, \alpha \Rightarrow \varphi}{\Gamma, \Delta, \alpha \leftrightarrow \beta \Rightarrow \varphi}\ (\leftrightarrow\Rightarrow)_2$$

Definition

We will call a sequent $\Gamma \Rightarrow \varphi$ in the signature $\{\wedge, \to\}$ *equivalential* if every formula occurring in it is either

(i) a variable,

(ii) recursively of the form $(\alpha \to \beta) \wedge (\beta \to \alpha)$,

that is, (i) and (ii) apply to α and β as well.

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Via the definition of $\leftrightarrow$, all rules of **GEq** are derivable in $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$.

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Hence, **GEq** is a definable fragment of $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$.

Lemma

*Let $\Gamma \Rightarrow \varphi$ be an equivalential sequent. If $\Gamma \Rightarrow \varphi$ is provable in **GEq**, then $\Gamma \Rightarrow \varphi$, rewritten in the language of $\mathbf{FL_{ew}}^{\{\wedge,\to\}}$, is provable in $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$.*

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Lemma

The rule $(\Rightarrow\leftrightarrow)$ *is invertible, that is, the sequent* $\Gamma \Rightarrow \alpha \leftrightarrow \beta$ *is provable in* **GEq** *if and only if the sequents* $\Gamma, \alpha \Rightarrow \beta$ *and* $\Gamma, \beta \Rightarrow \alpha$ *are both provable in* **GEq**.

An analogous result holds for $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$.

Lemma

The sequent $\Gamma \Rightarrow (\alpha \to \beta) \wedge (\beta \to \alpha)$ *is provable in* $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$ *if and only if the sequents* $\Gamma, \alpha \Rightarrow \beta$ *and* $\Gamma, \beta \Rightarrow \alpha$ *are provable in* $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$.

In general, an equivalential sequent $\Gamma \Rightarrow \varphi$ provable in $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$ will have non-equivalential sequents occurring in its proof.

Lemma

The sequent $\Gamma \Rightarrow \varphi$ is provable in **GEq** *if and only if finitely many sequents $\Gamma_1 \Rightarrow x_1, \ldots, \Gamma_n \Rightarrow x_n$, where $x_1, \ldots, x_n$ are variables, are provable in* **GEq**. *These sequents are effectively obtainable from $\Gamma \Rightarrow \varphi$.*

Definition

A formula $\alpha \to \beta$ will be called *pre-equivalential* if α and β are equivalential.

A multiset of formulas is *pre-equivalential* if all formulas occurring in it are pre-equivalential.

If Δ is a pre-equivalential multiset, then we define $\Delta^{\leftrightarrow} = \{\sigma \leftrightarrow \tau : \sigma \to \tau \in \Delta\}$.

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Lemma

Let $\Gamma, \Delta \Rightarrow \varphi$ *be a sequent such that* $\Gamma \cup \{\varphi\}$ *is equivalential and* Δ *is pre-equivalential. If* $\Gamma, \Delta \Rightarrow \varphi$ *is provable in* $\mathbf{GFL_{ew}}^{\{\wedge, \to\}}$, *then* $\Gamma, \Delta^{\leftrightarrow} \Rightarrow \varphi$ *is provable in* **GEq**.

Theorem

Let $\Gamma \Rightarrow \varphi$ be an equivalential sequent. Then $\Gamma \Rightarrow \varphi$ is provable in $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$ *if and only if $\Gamma \Rightarrow \varphi$ is provable in* **GEq***.*

Theorem

Let $\Gamma \Rightarrow \varphi$ be an equivalential sequent. Then $\Gamma \Rightarrow \varphi$ is provable in $\mathbf{GFL_{ew}}^{\{\wedge,\to\}}$ *if and only if $\Gamma \Rightarrow \varphi$ is provable in* **GEq**.

Proof

The backward direction is given by the first Lemma. For the forward direction, notice that $\Gamma \Rightarrow \varphi$ is $\Gamma, \Delta \Rightarrow \varphi$ of the previous Lemma, with $\Delta = \emptyset$, and thus the claim follows.

Ongoing work: find the axioms for the equivalential fragment.

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Similar techniques were developed by Kowalski to prove structural incompleteness of BCK, an implicational substructural logic.

Formula trees

In the following, we will identifiy each equivalential formula α with its parsing tree, with variables labelling the leaves.

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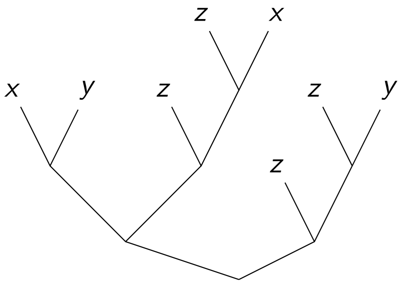
The formula $\Big((x \leftrightarrow y) \leftrightarrow (z \leftrightarrow (z \leftrightarrow x))\Big) \leftrightarrow (z \leftrightarrow (z \leftrightarrow y))$ is



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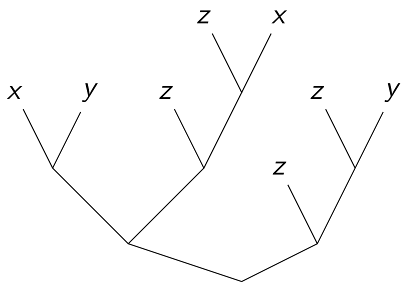

We will call such trees *formula trees*.

In a formula tree α each occurrence of a variable defines a unique path to the root. We denote the i-th occurrence of x (counting from the left) by $[x]_i$, and the path defined by $[x]_i$ by $\downarrow[x]_i$.

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Removing $\downarrow[x]_i$ from the formula tree α naturally produces a forest of formula trees: we will refer to it by $\alpha\downarrow[x]_i$.

Lemma

For any multiset Γ of formulas and for any formula α, the sequent $\Gamma \Rightarrow \alpha$ is provable in **GEq** *if and only if for every occurrence $[x]_i$ of every variable x occurring in α, the sequent $\Gamma, \alpha\downarrow[x]_i \Rightarrow x$ is provable in* **GEq**.

Next we deal with the provability of sequents of the form $\Gamma \Rightarrow x$, where x is a variable.

Lemma

Let Γ be a multiset of formulas and let x be a variable. The sequent $\Gamma \Rightarrow x$ is provable in **GEq** *if and only if either*

- $x \in \Gamma$ *or*
- *there exist a formula $\gamma \in \Gamma$, an occurrence $[x]_i$ in γ, and a partition P of $\Gamma \setminus \{\gamma\}$ such that for each $\beta \in \gamma{\downarrow}[x]_i$ there is a B which is either a partition class in P or empty, such that the sequent $B \Rightarrow \beta$ is provable in* **GEq**.

Final remarks

We have seen:

- Calculi for $\{\wedge, \to\}$-fragment and the equivalential fragment of $\mathbf{FL_{ew}}$;
- Properties of both, as cut elimination;
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- A strategy to simplify special provable sequents.

As future work:

- Try to find an axiomatization of the equivalential fragment of $\mathbf{FL_{ew}}$;
- Study structural completeness.

Thank you for your attention