Adjoints and the preservation of order

David Holgate

University of the Western Cape, South Africa

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(Together with Ana Belén Avilez and Bakulikira Claude Iragi)

What is required to "do topology"?

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{P}X \arrow[r, shift left, "f(-)"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left, "f^{-1}(-)"] \end{tikzcd}
\qquad
\begin{tikzcd} \mathbf{Top} \arrow[d, squiggly, no head] \\ \mathbf{Ord} \end{tikzcd}$$

Subspaces and continuity play out in **Ord** where for a given map $f : X \to Y$, $A \subseteq X$ and $B \subseteq Y$:

$$f(A) \subseteq B \Leftrightarrow A \subseteq f^{-1}(B).$$

Given any category $\mathcal{C}$

We require a pseudofunctor P such that...

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ PX \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & PY \arrow[l, shift left, "f_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[d, squiggly, no head] \\ \mathbf{Ord} \end{tikzcd}$$

...where for a given morphism $f : X \to Y$ we have an associated (covariant) Galois connection, such that for $a \in PX$ and $b \in PY$:

$$f^*(a) \leq b \Leftrightarrow a \leq f_*(b).$$

Topogenous orders on $\mathcal{C}$, for a given P

A topogenous order on $\mathcal{C}$ is a family $\triangleleft = \{\triangleleft_X \mid X \in \mathcal{C}\}$ of relations, each $\triangleleft_X$ on PX, such that for appropriate elements:

T1: $a \triangleleft b \Rightarrow a \leq b$,

T2: $a \leq b \triangleleft c \leq d \Rightarrow a \triangleleft d$,

T3: each $f : X \to Y$ is $\triangleleft$-continuous, $a \triangleleft_Y b \Rightarrow f_*(a) \triangleleft_X f_*(b)$.

Topogenous orders on $\mathcal{C}$, for a given P

A topogenous order on $\mathcal{C}$ is a family $\lhd = \{\lhd_X | \ X \in \mathcal{C}\}$ of relations, each $\lhd_X$ on PX, such that for appropriate elements:

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A few observations:

- Topogenous orders can be understood with the (topological) intuition that $a \lhd b$ iff b is a neighbourhood of a.
- Topogenous orders that respect infima – $a \lhd b$ for all $b \in B \Rightarrow a \lhd \bigwedge B$ – are equivalent to Dikranjan-Giuli closure operators.
- Topogenous orders that respect suprema – $a \lhd b$ for all $a \in A \Rightarrow \bigvee A \lhd b$ – are equivalent to interior operators.

An observation about continuity

Given a topogenous order on $\mathcal{C}$ and $f : X \to Y$ in $\mathcal{C}$, routine application of the properties of the orders renders that the following are equivalent (for suitable elements).

1. f is continuous: $a \lhd_Y b \Rightarrow f_*(a) \lhd_X f_*(b)$.
2. $f^*(a) \lhd_Y b \Rightarrow a \lhd_X f_*(b)$

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$(1) \Rightarrow (2)$: $f^*(a) \lhd b \Rightarrow a \leq f_* f^*(a) \lhd f_*(b) \Rightarrow a \lhd f_*(b)$

Galois connection

(T2)

$(2) \Rightarrow (1)$: $a \lhd b \Rightarrow f^* f_*(a) \leq a \lhd b \Rightarrow f^* f_*(a) \lhd b$

$\Rightarrow f_*(a) \lhd f_*(b)$

Strict morphisms

Let $\lhd$ be a topogenous order on $\mathcal{C}$. A map $f : X \to Y$ is called $\lhd$-strict if

$$f^*(a) \lhd_Y b \Leftrightarrow a \lhd_X f_*(b)$$

for $a \in PX$ and $b \in PY$.

The collection of $\lhd$-strict morphisms is well-behaved, containing the isomorphisms and closed under composition. Left and right cancellation properties often follow depending on the nature of P.

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$(\Rightarrow)$ $a \triangleleft b \Rightarrow a \triangleleft b \leq f_* f^*(b) \Rightarrow a \triangleleft f_* f^*(b) \Rightarrow f^*(a) \triangleleft f^*(b)$

← Galois connection ← (T2)

$(\Leftarrow)$ $a \triangleleft f_*(b) \Rightarrow f^*(a) \triangleleft f^* f_*(b) \leq b \Rightarrow f^*(a) \triangleleft b$

NB continuity is a given.

Some motivation from topology...

Consider the order $A \triangleleft B \Leftrightarrow \overline{A} \subseteq B$. For a continuous map $f : X \to Y$ we have the image/pre-image and pre-image/dual-image adjunctions, where $f_*(A) := Y \setminus f(X \setminus A)$.

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left, "f(-)"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left, "f^{-1}(-)"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, shift left, "f_*(-)"] \arrow[r, phantom, "\top"] & \mathcal{P}Y \arrow[l, shift left, "f^{-1}(-)"] \end{tikzcd}$$

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($\Rightarrow$) A is closed iff $A \lhd A$. Thus:

A closed $\Rightarrow$ $f(A) \lhd f(A)$ $\Rightarrow$ $f(A)$ closed.

($\Leftarrow$) If f is closed then $f(\overline{A}) = \overline{f(A)}$.

$\overline{A} \subseteq B \Rightarrow \overline{f(A)} = f(\overline{A}) \subseteq f(B)$

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($\Rightarrow$) U is open iff $X \setminus U \lhd X \setminus U$, whence

$f_*(X \setminus U) \lhd f_*(X \setminus U)$ giving

$\overline{Y \setminus f(U)} \subseteq Y \setminus f(U)$ and $f(U)$ open

($\Leftarrow$) Let f be open and $\overline{A} \subseteq B$. Then $f(X \setminus \overline{A})$ is open and:

$\overline{Y \setminus f(X \setminus A)} \subseteq Y \setminus f(X \setminus \overline{A}) \subseteq Y \setminus f(X \setminus B)$.

Locales and sublocales...

For a localic map $f : L \to M$ we have the localic image/pre-image adjunction between the coframes of sublocales.

$$\begin{tikzcd} \mathcal{S}(L) \arrow[r, shift left, "{f[-]}"] \arrow[r, phantom, "\perp"] & \mathcal{S}(M) \arrow[l, shift left, "{f_{-1}[-]}"] \end{tikzcd}$$

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For a sublocale S the *closure* and *interior* are, respectively, the smallest closed sublocale containing S and the largest open sublocale contained in S, and can be calculated by $\overline{S} = \mathfrak{c}(\bigwedge S)$ and $\mathrm{S}^\circ = \mathfrak{o}(\bigvee\{\mathrm{a} \mid \mathfrak{o}(\mathrm{a}) \subseteq \mathrm{S}\})$.

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This gives us two orders to consider and the associated $\triangleleft$-strict maps.

$$S \triangleleft T \Leftrightarrow \overline{S} \subseteq T \qquad\qquad S \triangleleft T \Leftrightarrow S \subseteq T^\circ$$

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Topology and the open set lattice

Again consider the order $A \lhd B \Leftrightarrow \overline{A} \subseteq B$, but now restricted to open sets. For a continuous map $f : X \to Y$ we have the adjunction with $f_*(A) := Y \setminus \overline{f(X \setminus A)}$.

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Closed: Take any constant map $f : \mathbb{R} \to \mathbb{R}$ on the reals with the usual topology.

$f(x) = 0 \;\; \forall x \in \mathbb{R}$: $f_*(A) = \mathbb{R} \setminus \{0\}$ for any open $A \subseteq \mathbb{R}$ $(A \neq \emptyset)$ and so $\overline{f_*(A)} = \mathbb{R} \nsubseteq f_*(B)$ whenever $\overline{A} \subseteq B$.

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$\overline{f_*(A)} = \mathbb{R} \nsubseteq f_*(B)$ whenever $\bar{A} \subseteq B$.

Open: Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \arctan(x)$.

$\mathbb{R} \lhd \mathbb{R}$ but $\overline{f_*(\mathbb{R})} = \overline{\mathbb{R} - \overline{(-\pi/2, \pi/2)}} \nsubseteq \mathbb{R} \setminus \overline{(-\pi/2, \pi/2)}$

($\mathbb{R}$ is a well-behaved space.)

(In general $\emptyset \lhd \emptyset \Rightarrow f_*(\emptyset) \lhd f_*(\emptyset) \Rightarrow \overline{Y \setminus \overline{f(X)}} \subseteq Y \setminus \overline{f(X)} \Rightarrow \overline{f(X)}$ is open.)

Topology and the open set lattice...

A continuous map $f : X \to Y$ is weakly open if for any $A \subseteq Y$, $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$.

Proposition

If a continuous map f is both closed and weakly open, then f is $\lhd$-strict.

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Proposition

If a continuous map f is both closed and weakly open, then f is $\triangleleft$-strict.

Note: If f is closed then $f_*(A) = Y \setminus \overline{f(X \setminus A)} = Y \setminus f(X \setminus A)$ for A open.

Let $\overline{A} \leq B$, then:

$$f^{-1}(\overline{f_*(A)}) \overset{\text{weakly open}}{=} \overline{f^{-1}(f_*(A))} \overset{\text{Galois connection}}{\leq} \overline{A} \leq B$$

Note: Can't use the adjunction since $\overline{f_*(A)}$ not open.

$$\Rightarrow X \setminus B \leq X \setminus f^{-1}(\overline{f_*(A)}) = f^{-1}(Y \setminus \overline{f_*(A)})$$

$$\Rightarrow f(X \setminus B) \leq Y \setminus \overline{f_*(A)}$$

$$\Rightarrow \overline{f_*(A)} \leq Y \setminus f(X \setminus B) = f_*(B).$$

Frames and frame homomomorphisms

Any frame homomorphism $h : L \to M$ has a right adjoint h_*.

$$\begin{tikzcd} L \arrow[r, shift left=2, "h"] \arrow[r, phantom, "\perp"] & M \arrow[l, shift left=2, "h_*"] \end{tikzcd}$$

We say that h is weakly open if h preserves pseudocomplements, $h(a^*) = h(a)^*$ for any $a \in L$, while h is closed if $h_*(h(a) \vee b) = a \vee h_*(b)$ for $a \in L$ and $b \in M$.

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$$
\begin{tikzcd}
L \arrow[r, shift left=1.5ex, "h"] \arrow[r, phantom, "\perp"] & M \arrow[l, shift left=1.5ex, "h_*"]
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The rather below relation: $a \prec b \Leftrightarrow a^* \vee b = 1$ is a topogenous order.

Proposition

If h is both closed and weakly open, then h is $\prec$-strict.

$$\begin{aligned} h(x) \prec y &\Leftrightarrow h(x)^* \vee y = 1 \\ &\Leftrightarrow h(x^*) \vee y = 1 \\ &\Rightarrow h_*(h(x^*) \vee y) = h_*(1) \\ &\Leftrightarrow x \vee h_*(y) = 1 \\ &\Leftrightarrow x \prec h_*(y) \end{aligned}$$

Other examples

- For the completely below relation, the $\prec\!\!\prec$-strict frame homomorphisms were termed assertive in [3]. Clearly $\prec$-strict $\Rightarrow \prec\!\!\prec$-strict.
- For the way below relation, $\ll$-strict maps are called perfect (with care taken regarding the adjunction).
- Many examples arise from situations where the $\triangleleft$-fixed elements $\{a \mid a \triangleleft a\}$ characterise $a \triangleleft b \Leftrightarrow a \leq c \leq b$ for some $\triangleleft$-fixed c. In these cases f is $\triangleleft$-strict iff f^* preserves $\triangleleft$-fixed elements.
- For $\bigwedge$-respecting orders and the associated closure operator, $c(a) := \bigwedge\{b \mid a \triangleleft b\}$, $\triangleleft$-strict coincides with closed.
- For $\bigvee$-respecting orders and the associated interior operator, $i(b) := \bigvee\{a \mid a \triangleleft b\}$, $\triangleleft$-strict coincides with open.

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