Lifting structure from states to propositions
the various Day extensions available

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Joint work with Edmund Robinson

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Setting

We focus on **2**-enriched categories, i.e. preorders.

Let $\mathbb{C}$ be a set of states/contexts.

Thus, the propositions on $\mathbb{C}$ can be equated with (co-)presheaves $[\mathbb{C} \to \mathbf{2}]$.

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Via **Day convolution**, a monoidal product $\otimes$ on $\mathbb{C}$ *lifts* to a new logical connective:

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Can we lift arbitrary operations on $\mathbb{C}$ to connectives on $[\mathbb{C} \to \mathbf{2}]$?

Profunctors

Definition

A *profunctor* $\mathbb{A} \nrightarrow \mathbb{B}$ is a functor $\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \to \mathbf{2}$.

Profunctors $\mathbb{A} \overset{F}{\nrightarrow} \mathbb{B}$ and $\mathbb{B} \overset{G}{\nrightarrow} \mathbb{C}$ compose via the coend

$$G \circ F(c, a) = \int^{b \in \mathbb{B}} G(c, b) \times F(b, a),$$

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Examples

- For a functor $F \colon \mathbb{A} \to \mathbb{B}$,

$$\mathbb{A} \xrightarrow{\mathbb{B}(1,F)} \mathbb{B}, \quad \mathbb{B} \xrightarrow{\mathbb{B}(F,1)} \mathbb{A}.$$

- $\mathbf{Prof}(\mathbb{C}, \mathbf{1}) \cong [\mathbb{C} \to \mathbf{2}]$.

Day extension I

Definition 1 (Robinson + W.)

Given an n-ary operation $\theta\colon \mathbb{C}^n \to \mathbb{C}$, the *Day extension*

$$\theta_D\colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$$

acts by sending $\varphi_1, \ldots, \varphi_n \in [\mathbb{C} \to \mathbf{2}]$ to the composite

$$\mathbb{C} \xrightarrow{\mathbb{C}(\theta,1)} \mathbb{C}^n \xrightarrow{\varphi_1 \times \cdots \times \varphi_n} \mathbf{1}.$$

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Alternatively, this is the co-end:

$$\theta_D(\varphi_1, \ldots, \varphi_n)(c) = \int^{c_1,\ldots,c_n \in \mathbb{C}^n} \varphi_1(c_1) \times \cdots \times \varphi_n(c_n) \times \mathbb{C}(\theta(c_1, \ldots, c_n), c),$$

or the left Kan extension:
$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\theta"] \arrow[d, "\langle \varphi_1,\ldots,\varphi_n \rangle"'] & \mathbb{C} \arrow[d, dashed] \\ \mathbf{2}^n \arrow[r, "\prod"] & \mathbf{2}. \end{tikzcd}$$

Operads

Definition

A **CAT**-operad P consists of:

(1) a category $P(n)$ of n-ary operations, for each $n \in \mathbb{N}$,

(2) a *unit* $1 \in P(1)$,

(3) and for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a *multi-composition* functor

$$\circ \colon P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n),$$

satisfying the equations:

$$\theta \circ (1, \ldots, 1) = \theta = 1 \circ \theta$$

and $\quad \theta \circ (\theta_1 \circ (\theta_1^1, \ldots, \theta_1^{k_1}), \ldots, \theta_n \circ (\theta_n^1, \ldots, \theta_n^{k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_1^1, \ldots, \theta_n^{k_n}).$

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Example

For any $\mathbb{C}$, taking endomorphisms $\mathrm{End}(\mathbb{C})(n) = [\mathbb{C}^n \to \mathbb{C}]$ defines an operad.

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Definition

A *pseudo-morphism* of **CAT**-operads $\omega \colon P \to Q$ consists of functors $\omega_n \colon P(n) \to Q(n)$, for each $n \in \mathbb{N}$, and coherent isomorphisms

$$\omega_1(1) \cong 1, \qquad \omega_n(\theta) \circ (\omega_{k_1}(\theta_1), \ldots, \omega_{k_n}(\theta_n)) \cong \omega_{k_1+\cdots+k_n}(\theta \circ (\theta_1, \ldots, \theta_n)).$$

Day extension II

Theorem 1 (Robinson + W.)

The Day extension defines a pseudo-morphism of **CAT**-operads

$$(-)_D \colon \mathsf{End}(\mathbb{C}) \to \mathsf{End}([\mathbb{C} \to \mathbf{2}])^{\mathsf{op}}.$$

Equations are sent to pseudo-equations.

Day extension III

An n-ary partial operation $\theta \colon \mathbb{C}^n \rightharpoondown \mathbb{C}$ is a span

$$\mathbb{C}^n \xhookleftarrow{i} \mathbb{D} \xrightarrow{\theta} \mathbb{C}$$

where i is *full*.

Day extension III

An n-ary partial operation $\theta\colon \mathbb{C}^n \rightharpoondown \mathbb{C}$ is a span

$$\mathbb{C}^n \xleftarrow{\;i\;} \mathbb{D} \xrightarrow{\;\theta\;} \mathbb{C}$$

where i is *full*.

Definition 2 (Robinson + W.)

Given an n-ary partial operation $\theta\colon \mathbb{C}^n \rightharpoondown \mathbb{C}$, the *Day extension* is the *total* operation

$$\theta_D\colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$$

that acts by sending $\varphi_1, \ldots, \varphi_n \in [\mathbb{C} \to \mathbf{2}]$ to the composite

$$\mathbb{C} \xrightarrow{\mathbb{C}(\theta,1)} \mathbb{D} \xrightarrow{\mathbb{C}^n(1,i)} \mathbb{C}^n \xrightarrow{\varphi_1\times\cdots\times\varphi_n} \mathbf{1}.$$

Partial operads

Definition

A partial **CAT**-operad P consists of:

(1) a category $P(n)$ of n-ary operations, for each $n \in \mathbb{N}$,

(2) a *unit* $1 \in P(1)$,

(3) and for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a *multi-composition* partial functor

$$\circ \colon P(n) \times P(k_1) \times \cdots \times P(k_n) \rightharpoonup P(k_1 + \cdots + k_n),$$

satisfying the equations:

$$\theta \circ (1, \ldots, 1) = \theta = 1 \circ \theta$$

and $\quad \theta \circ (\theta_1 \circ (\theta_1^1, \ldots, \theta_1^{k_1}), \ldots, \theta_n \circ (\theta_n^1, \ldots, \theta_n^{k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_1^1, \ldots, \theta_n^{k_n})$

whenever sensible.

Day extension IV

Theorem 2 (Robinson + W.)

The Day extension for partial operations defines a pseudo-morphism of partial **CAT**-operads

$$(-)_D \colon \mathrm{End}(\mathbb{C})_{\mathrm{par}} \to \mathrm{End}([\mathbb{C} \to \mathbf{2}])^{\mathrm{op}}.$$

Strong equality between partial terms is pseudo-preserved.

Sketch proof

Suppose $\theta \circ (\theta_1, \ldots, \theta_n)$ exists, then:

$$
\begin{tikzcd}
 & \mathbb{D} \arrow[r, "\theta"] \arrow[d, hook, "i"'] & \mathbb{C} \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[ur, dashed] \arrow[r, "{(\theta_1,\ldots,\theta_n)}"'] \arrow[d, hook, "{(i_1,\ldots,i_n)}"'] & \mathbb{C}^n & \\
\mathbb{C}^{k_1+\cdots+k_n} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{D} \arrow[dl, "\shortmid"{marking}] \arrow[d, "\shortmid"{marking}, "{\mathbb{C}(1,i)}"] \arrow[dl, phantom, shift right=6ex, "\cong"] & \mathbb{C} \arrow[l, "\shortmid"{marking}, "{\mathbb{C}(\theta,1)}"'] \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[d, "\shortmid"{marking}, "{\mathbb{C}^{k_1+\cdots+k_n}(1,(i_1,\ldots,i_n))}"'] & \mathbb{C}^n \arrow[l, "\shortmid"{marking}, "{\mathbb{C}^n((\theta_1,\ldots,\theta_n),1)}"] & \\
\mathbb{C}^{k_1+\cdots+k_n} \arrow[d, "\shortmid"{marking}, "{\varphi_1^1\times\cdots\times\varphi_n^{k_n}}"'] & & \\
\mathbf{1} & &
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\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{D} \arrow[dl, "|" marking] \arrow[d, "|" marking, "\mathbb{C}(1,i)"] \arrow[dl, phantom, shift left=4, "\cong"] & \mathbb{C} \arrow[l, "|" marking, "\mathbb{C}(\theta,1)"'] \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[d, "|" marking, "\mathbb{C}^{k_1+\cdots+k_n}(1,(i_1,\ldots,i_n))"'] & \mathbb{C}^n \arrow[l, "|" marking, "\mathbb{C}^n((\theta_1,\ldots,\theta_n),1)"] & \\
\mathbb{C}^{k_1+\cdots+k_n} \arrow[d, "|" marking, "\varphi_1^1 \times \cdots \times \varphi_n^{k_n}"'] & & \\
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$$

which yields

$$(\theta \circ (\theta_1, \ldots, \theta_n))_D(\varphi_1^1, \ldots, \varphi_n^{k_n}) \cong \theta_D \circ (\theta_{1D}, \ldots, \theta_{nD})(\varphi_1^1, \ldots, \varphi_n^{k_n})$$

as desired.

Residuation

Proposition

For any $\theta \colon \mathbb{C}^n \to \mathbb{C}$, the Day extension $\theta_D \colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$ is *residuated* in each variable.

That is, there is an operation $R^j_\theta \colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$ such that

$$[\mathbb{C} \to 2](\theta_D(\varphi_1, \ldots, \varphi_n), \psi) \cong [\mathbb{C} \to 2](\varphi_j, R^j_\theta(\psi, \varphi_1, \ldots, \varphi_{j-1}, \varphi_{j+1}, \ldots, \varphi_n)).$$

The proof is given by taken $R^j_\theta(\psi, \varphi_1, \ldots, \varphi_{j-1}, \varphi_{j+1}, \ldots, \varphi_n)(a)$ to be the *end*:

$$\int_{\vec{c} \in \mathbb{C}^n,\, \vec{d} \in \mathbb{D}} \left[\left(\mathbb{C}^n(\vec{c}[a/c_j], \vec{d}) \times \mathbb{C}(\theta(\vec{d}, c_j), c_j) \times \prod_{i \neq j} \varphi_i(c_i) \right) \to \psi(c_j) \right].$$

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That is, there is an operation $R^j_\theta\colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$ such that

$$\theta_D(\varphi_1, \dots, \varphi_n) \leqslant \psi \iff \varphi_j \leqslant R^j_\theta(\psi, \varphi_1, \dots, \varphi_{j-1}, \varphi_{j+1}, \dots, \varphi_n).$$

The proof is given by taken $R^j_\theta(\psi, \varphi_1, \dots, \varphi_{j-1}, \varphi_{j+1}, \dots, \varphi_n)(a)$ to be the *end*:

$$\int_{\vec{c}\in\mathbb{C}^n,\, \vec{d}\in\mathbb{D}} \left[\left(\mathbb{C}^n(\vec{c}[a/c_j], \vec{d}) \times \mathbb{C}(\theta(\vec{d}, c_j), c_j) \times \prod_{i\neq j} \varphi_i(c_i)\right) \to \psi(c_j)\right].$$

Composition via pullbacks?

We no longer require the domain to be full for a partial operation $\mathbb{C}^n \xhookleftarrow{i} \mathbb{D} \xrightarrow{\theta} \mathbb{C}$.

Composition via pullbacks?

We no longer require the domain to be full for a partial operation $\mathbb{C}^n \xhookleftarrow{i} \mathbb{D} \xrightarrow{\theta} \mathbb{C}$.

Why not define the composite $\theta \circ^{\mathsf{pb}} (\theta_1, \ldots, \theta_n)$ using pullbacks?

That is, define $\theta \circ^{\mathsf{pb}} (\theta_1, \ldots, \theta_n)$ as the outside span in:

$$
\begin{tikzcd}
& \mathbb{A} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D} \arrow[r, "\theta"] \arrow[d, hook, "i"] & \mathbb{C} \\
& \mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[r, "(\theta_1, \ldots, \theta_n)"'] \arrow[d, hook, "(i_1, \ldots, i_n)"'] & \mathbb{C}^n & \\
& \mathbb{C}^{k_1 + \cdots + k_n} & &
\end{tikzcd}
$$

This yields a (total) **CAT**-operad $\mathrm{End}(\mathbb{C})^{\mathsf{pb}}_{\mathsf{par}}$.

Exact squares

Pullback squares are *not* exact.

Definition (Guitart)

A commuting square is *exact* if the *mate* between profunctors is an isomorphism.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathbb{B} \arrow[d, "H"] \\
\mathbb{C} \arrow[r, "K"'] & \mathbb{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[d, "\mathbb{C}(1,G)"'] \arrow[dr, Rightarrow, "\sim"] & \mathbb{B} \arrow[l, "\mathbb{B}(F,1)"', "|" marking] \arrow[d, "\mathbb{D}(1,H)", "-" marking] \\
\mathbb{C} & \mathbb{D} \arrow[l, "\mathbb{D}(K,1)", "|" marking]
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\mathbb{A} \arrow[d, "\mathbb{C}(1{,}G)"'] \arrow[dr, Rightarrow, "\sim"] & \mathbb{B} \arrow[l, "\mathbb{B}(F{,}1)"', "|" marking] \arrow[d, "\mathbb{D}(1{,}H)", "|" marking] \\
\mathbb{C} & \mathbb{D} \arrow[l, "\mathbb{D}(K{,}1)", "|" marking]
\end{tikzcd}
$$

The diagrams below yield a morphism $(\theta \circ^{\mathrm{pb}} (\theta_1, \ldots, \theta_n))_D \to \theta_D \circ (\theta_{1D}, \ldots, \theta_{nD})$.

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\lrcorner" very near start] & & \mathbb{D} \arrow[r, "\theta"] \arrow[d, hook, "i"] & \mathbb{C} \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[rr, "(\theta_1,\ldots,\theta_n)"'] \arrow[d, hook, "(i_1,\ldots,i_n)"'] & & \mathbb{C}^n & \\
\mathbb{C}^{k_1+\cdots+k_n} & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[d, "|" marking] \arrow[drr, Rightarrow] & & \mathbb{D} \arrow[ll, "|" marking] \arrow[d, "\mathbb{C}^n(1{,}i)", "|" marking] & \mathbb{C} \arrow[l, "\mathbb{C}(\theta{,}1)"', "|" marking] \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[d, "\mathbb{C}^{k_1+\cdots+k_n}(1{,}(i_1{,}\ldots{,}i_n))"', "|" marking] & & \mathbb{C}^n \arrow[ll, "\mathbb{C}^n((\theta_1{,}\ldots{,}\theta_b){,}1)", "|" marking] & \\
\mathbb{C}^{k_1+\cdots+k_n} & & &
\end{tikzcd}
$$

Thus, $(-)_D \colon \mathrm{End}(\mathbb{C})^{\mathrm{pb}}_{\mathrm{par}} \to \mathrm{End}([\mathbb{C} \to \mathbf{2}])^{\mathrm{op}}$ is only a *lax* morphism of **CAT**-operads.

Day extension V

Comma squares *are* exact.

Define a multi-composite on n-ary spans

$$\mathbb{C}^n \xleftarrow{G} \mathbb{D} \xrightarrow{F} \mathbb{C}$$

via the outside span in the comma square

$$\begin{tikzcd}
G{\downarrow}(F_1, \ldots, F_n) \arrow[r] \arrow[d] & \mathbb{D} \arrow[r, "F"] \arrow[d, "G"] & \mathbb{C} \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[r, "(F_1,\ldots,F_n)"'] \arrow[d, "(G_1,\ldots,G_n)"'] & \mathbb{C}^n & \\
\mathbb{C}^{k_1+\cdots+k_n} & &
\end{tikzcd}$$

yielding a **CAT**-operad $\mathrm{Span}(\mathbb{C})$.

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\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[r, "{(F_1,\ldots,F_n)}"'] \arrow[d, "{(G_1,\ldots,G_n)}"'] & \mathbb{C}^n & \\
\mathbb{C}^{k_1+\cdots+k_n} & &
\end{tikzcd}$$

yielding a **CAT**-operad $\mathrm{Span}(\mathbb{C})$.

Define $F_D(\varphi_1, \ldots, \varphi_n)$ as the composite

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "{\mathbb{C}(F,1)}", "|" marking] & \mathbb{D} \arrow[r, "{\mathbb{C}^n(1,G)}", "|" marking] & \mathbb{C}^n \arrow[r, "\varphi_1 \times \cdots \times \varphi_n", "|" marking] & \mathbf{1}.
\end{tikzcd}$$

Then the diagram of profunctors pseudo-commutes

$$\begin{tikzcd}
G{\downarrow}(F_1, \ldots, F_n) \arrow[d, "|" marking] & \mathbb{D} \arrow[l, "|" marking] \arrow[d, "{\mathbb{C}^n(1,G)}", "|" marking] \arrow[dl, phantom, "\cong"] & \mathbb{C} \arrow[l, "{\mathbb{C}(F,1)}"', "|" marking] \\
\mathbb{D}_1 \times \cdots \times \mathbb{D}_n \arrow[d, "{\mathbb{C}^{k_1+\cdots+k_n}(1,(G_1,\ldots,G_n))}"'] & \mathbb{C}^n \arrow[l, "{\mathbb{C}^n((F_1,\ldots,F_n),1)}", "|" marking] & \\
\mathbb{C}^{k_1+\cdots+k_n} & &
\end{tikzcd}$$

and so $(F \circ (F_1, \ldots, F_n))_D \cong F_D \circ (F_{1D}, \ldots, F_{nD})$.

Thus, we obtain a pseudo-morphism $(-)_D \colon \mathrm{Span}(\mathbb{C}) \to \mathrm{End}([\mathbb{C} \to \mathbf{2}]^{\mathrm{op}}$.

Thank you for your attention

Main references

[1] Day, B. (1970). On closed categories of functors. In: MacLane, S. et al., (eds.) *Reports of the Midwest Category Seminar* IV, Berlin Heidelberg, Springer, 1–38.

[2] Day, B. (1973). On closed categories of functors II. In: *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, Springer, 20–54.

[3] Guitart, R. (1980). Relations et carrés exacts. *Annales Des Sciences Mathématiques Du Québec* IV(2), 103–125.

[4] Robinson, E. and Wrigley, J. (2026). Day algebras. *Mathematical Structures in Computer Science* 36(e6), 1–22.