Localic Relations with Open Cones

Nesta van der Schaaf

Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, Laboratoire Méthodes Formelles

https://nvds.site/

arXiv:2605.04038

28 July

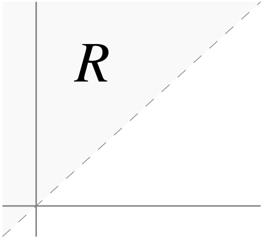
TACL 2026

Localic relations

Locallic relations

spatial
relation

$$r\colon R \subseteq S \times S$$



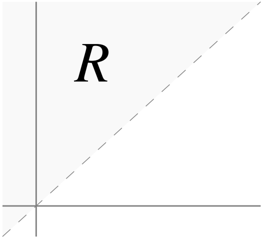
localic
relation

$$r\colon R \rightarrowtail X \times X$$

Localic relations

spatial relation

$$r\colon R \subseteq S \times S$$



localic relation

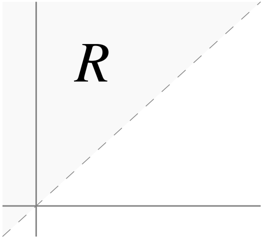
$$r\colon R \rightarrowtail X \times X$$

sublocale

Localic relations

spatial relation

$$r\colon R \subseteq S \times S$$



localic relation

$$r\colon R \rightarrowtail X \times X$$

sublocale

product locale

Localic relations

spatial
relation

$r \colon R \subseteq S \times S$

$\updownarrow$

$\uparrow, \downarrow \colon\ \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$

localic
relation

$r \colon R \rightarrowtail X \times X$

sublocale

product
locale

Localic relations

spatial relation

$$r\colon R \subseteq S \times S$$

$$xRy \iff x \in {\downarrow}\{y\}$$

$$\uparrow, \downarrow\colon\ \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$$

localic relation

$$r\colon R \rightarrowtail X \times X$$

sublocale

product locale

Localic relations

spatial
relation

$r\colon R \subseteq S \times S$

$xRy \iff x \in {\downarrow}\{y\}$

$\uparrow, \downarrow\colon\ \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$

localic
relation

$r\colon R \rightarrowtail X \times X$

sublocale

product
locale

$\boxed{\Uparrow, \Downarrow\colon\ \mathcal{O}X \xrightarrow{\ ?\ } \mathcal{O}X}$

Plan

localic
relation

$$r\colon R \rightarrowtail X \times X$$

Plan

localic
relation

$r\colon R \rightarrowtail X \times X$

induced
cones

$\Uparrow, \Downarrow\colon \mathcal{O}X \longrightarrow \mathcal{O}X$

Plan

localic relation

$r\colon R \rightarrowtail X \times X$

induced cones

$\Uparrow, \Downarrow\colon\ \mathcal{O}X \longrightarrow \mathcal{O}X$

cones

$\triangle, \triangledown\colon\ \mathcal{O}X \longrightarrow \mathcal{O}X$

Plan

localic relation

$r\colon R \rightarrowtail X \times X$

induced cones

$\Uparrow, \Downarrow\colon\ \mathcal{O}X \longrightarrow \mathcal{O}X$

induced relation

$r_{\boxslash}\colon R_{\boxslash} \rightarrowtail X \times X$

cones

$\triangle, \triangledown\colon\ \mathcal{O}X \longrightarrow \mathcal{O}X$

Plan

localic relation

$r\colon R \rightarrowtail X \times X$

induced cones

$\Uparrow, \Downarrow\colon \mathcal{O}X \longrightarrow \mathcal{O}X$

$$\boxed{\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \Delta\!\!\!\!\nabla\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}}$$

induced relation

$r_{\Delta\!\!\!\nabla}\colon R_{\Delta\!\!\!\nabla} \rightarrowtail X \times X$

cones

$\Delta, \nabla\colon \mathcal{O}X \longrightarrow \mathcal{O}X$

Plan

localic relation

$r \colon R \rightarrowtail X \times X$

induced cones

$\Uparrow, \Downarrow \colon \mathcal{O}X \longrightarrow \mathcal{O}X$

$$\begin{tikzcd} \boxed{\mathbf{rocLoc}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \boxslash\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

induced relation

$r_{\boxslash} \colon R_{\boxslash} \rightarrowtail X \times X$

cones

$\Delta, \nabla \colon \mathcal{O}X \longrightarrow \mathcal{O}X$

Open cones

relation

$$r = (s,t) \colon R \rightarrowtail X \times X$$

[Koc89] A. Kock, *“A Godement Theorem for Locales”*, 1989.
[KM92] A. Kock et al., *“Every Étendue Comes from a Local Equivalence Relation”*, 1992.

Open cones

relation

$$r = (s, t) \colon R \rightarrowtail X \times X$$

Definition. R has *open cones* if $s, t \colon R \to X$ are open.

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.
[KM92] A. Kock et al., *"Every Étendue Comes from a Local Equivalence Relation"*, 1992.

Open cones

relation

$$r = (s, t) \colon R \rightarrowtail X \times X$$

Definition. R has *open cones*
if $s, t \colon R \to X$ are open.

$s_! \dashv s^{-1}$ and $t_! \dashv t^{-1}$
with Frobenius reciprocity

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.
[KM92] A. Kock et al., *"Every Étendue Comes from a Local Equivalence Relation"*, 1992.

Open cones

relation

$$r = (s,t)\colon R \rightarrowtail X \times X$$

Definition. R has *open cones* if $s,t\colon R \to X$ are open.

$s_! \dashv s^{-1}$ and $t_! \dashv t^{-1}$
with Frobenius reciprocity

Spatially. R has oc iff $\uparrow, \downarrow$ preserve open subsets.

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.
[KM92] A. Kock et al., *"Every Étendue Comes from a Local Equivalence Relation"*, 1992.

Open cones

relation

$$r = (s,t)\colon R \rightarrowtail X \times X$$

Definition. R has *open cones* if $s, t\colon R \to X$ are open.

$s_! \dashv s^{-1}$ and $t_! \dashv t^{-1}$
with Frobenius reciprocity

Spatially. R has oc iff $\uparrow, \downarrow$ preserve open subsets.

e.g. $(\mathbb{R}, \leqslant)$, interval topology, ordered vector spaces, spacetimes

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.
[KM92] A. Kock et al., *"Every Étendue Comes from a Local Equivalence Relation"*, 1992.

The category of open cone relations

Definition. Define a category with objects (X, R): locales X equipped with oc R

$$\boxed{\mathbf{rocLoc}}$$

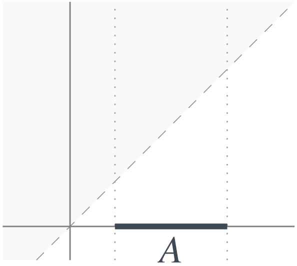
and morphisms

$$f\colon (X,R) \longrightarrow (Y,Q)$$

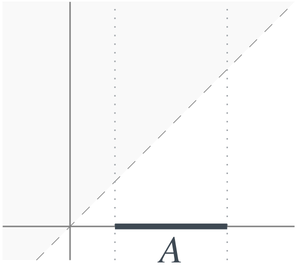
the *monotone* locale maps $f\colon X \to Y$:

$$(f \times f)[R] \subseteq Q$$

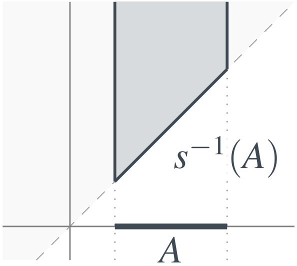
$R \longmapsto \Uparrow, \Downarrow$



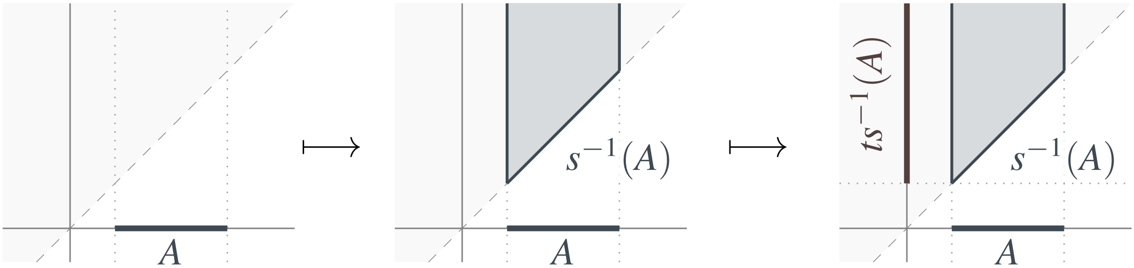
$$R \longmapsto \uparrow, \downarrow$$



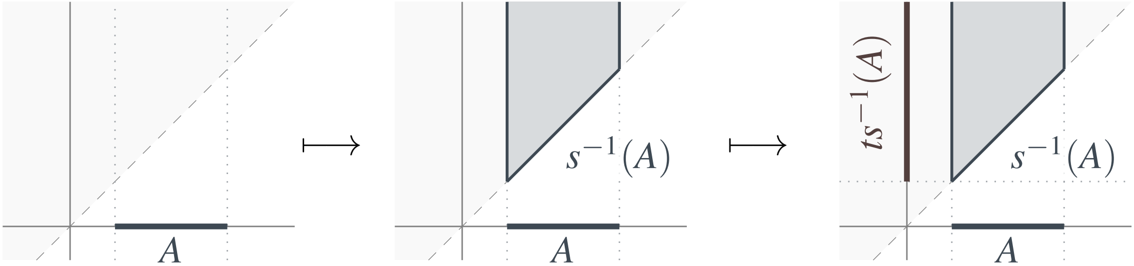
$\longmapsto$



$R \longmapsto \Uparrow, \Downarrow$

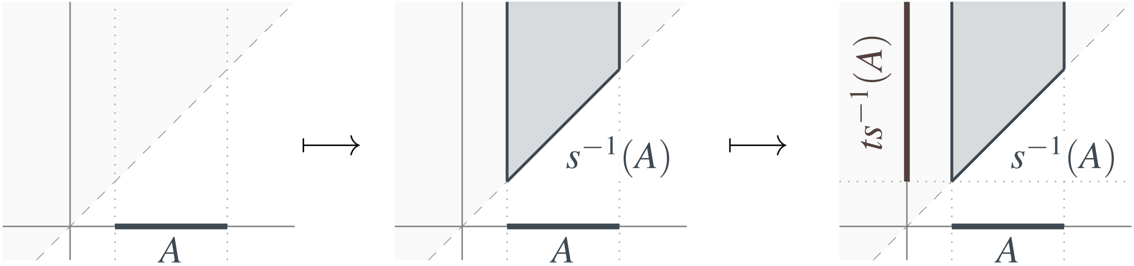


$R \longmapsto \uparrow, \downarrow$



$$\boxed{\uparrow = ts^{-1} \qquad \downarrow = st^{-1}}$$

$R \longmapsto \Uparrow, \Downarrow$



$$\boxed{\uparrow = ts^{-1} \qquad \downarrow = st^{-1}}$$

$$\boxed{\begin{gathered} \Uparrow := t_! s^{-1} \qquad \Downarrow := s_! t^{-1} \\ \mathcal{O}X \xrightarrow{s^{-1}} \mathcal{O}R \xrightarrow{t_!} \mathcal{O}X \end{gathered}}$$

Conic frames?

$$\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left, ""] \arrow[r, phantom, "\perp"] & \boxed{\triangledown\mathbf{Frm}^{\mathrm{op}}} \arrow[l, shift left, ""] \end{tikzcd}$$

Conic frames?

$$\boxed{\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \boxed{\nabla\!\!\!\!\triangle\mathbf{Frm}^{\mathrm{op}}} \arrow[l, shift left] \end{tikzcd}}$$

What should $(L, \triangle, \nabla) \in \nabla\!\!\!\!\triangle\mathbf{Frm}$ be?

Conic frames?

$$\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \boxed{\nabla\!\!\!\!\triangle\mathbf{Frm}^{\mathrm{op}}} \arrow[l, shift left] \end{tikzcd}$$

What should $(L, \triangle, \nabla) \in \nabla\!\!\!\!\triangle\mathbf{Frm}$ be?

$\Uparrow, \Downarrow$ preserve joins, since $s_!, s^{-1}, t_!, t^{-1}$ do

Parallelness

When do $\triangle$, ∇ come from *one* relation?

[JT51] B. Jónsson et al., *“Boolean Algebras with Operators. Part I”*, 1951.

Parallelness

When do $\triangle$, $\triangledown$ come from *one* relation?

Spatially:

$x \in {\downarrow}\{y\} \iff y \in {\uparrow}\{x\}$

[JT51] B. Jónsson et al., *“Boolean Algebras with Operators. Part I”*, 1951.

Parallelness

When do $\vartriangle$, $\triangledown$ come from *one* relation?

Spatially:

$$x \in {\downarrow}\{y\} \iff y \in {\uparrow}\{x\}$$

Definition. $\vartriangle$, $\triangledown$ are *parallel* if: (cf. [JT51])

$$\vartriangle x \wedge y \leqslant \vartriangle(x \wedge \triangledown y) \quad \text{and} \quad x \wedge \triangledown y \leqslant \triangledown(\vartriangle x \wedge y).$$

[JT51] B. Jónsson et al., *"Boolean Algebras with Operators. Part I"*, 1951.

Parallelness

When do $\triangle, \triangledown$ come from *one* relation?

Spatially:

$x \in {\downarrow}\{y\} \iff y \in {\uparrow}\{x\}$

Lemma. If $\triangle, \triangledown$ are parallel, then: (cf. [JT51])

$$x \wedge \triangledown y = \bot \iff \triangle x \wedge y = \bot.$$

[JT51] B. Jónsson et al., *"Boolean Algebras with Operators. Part I"*, 1951.

Parallelness

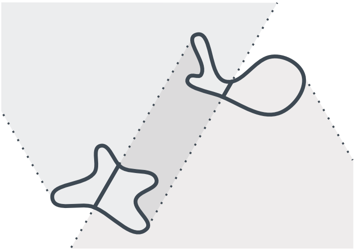
When do Δ, ∇ come from *one* relation?

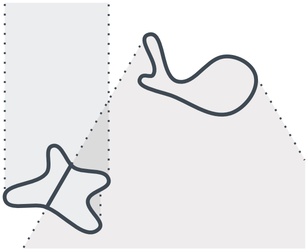
Spatially:

$$x \in {\downarrow}\{y\} \iff y \in {\uparrow}\{x\}$$

Lemma. If Δ, ∇ are parallel, then: (cf. [JT51])

$$x \wedge \nabla y = \bot \iff \Delta x \wedge y = \bot .$$





[JT51] B. Jónsson et al., *"Boolean Algebras with Operators. Part I"*, 1951.

Conic frames

Definition. A *conic frame* (L, Δ, ∇) is a frame L equipped with join-preserving and parallel $\Delta, \nabla \colon L \to L$.

$$\boxed{\Delta\!\!\!\nabla\mathbf{Frm}}$$

Conic frames

Definition. A *conic frame* $(L, \triangle, \nabla)$ is a frame L equipped with join-preserving and parallel $\triangle, \nabla\colon L \to L$.

$$\boxed{\triangle\!\!\!\nabla\mathbf{Frm}}$$

A *conic morphism*

$$h\colon (L, \triangle, \nabla) \longrightarrow (M, \underline{\triangle}, \overline{\nabla})$$

is a frame map $h\colon L \to M$:

$$\underline{\triangle} \circ h \leqslant h \circ \triangle,$$

$$\overline{\nabla} \circ h \leqslant h \circ \nabla\,.$$

Conic frames

$$\begin{aligned} \mathsf{cone}\colon \mathbf{rocLoc} &\longrightarrow \boxslash\mathbf{Frm}^{\mathrm{op}} \\ (X, R) &\longmapsto \big(\mathcal{O}X, \Uparrow, \Downarrow\big) \\ f &\longmapsto f^{-1} \end{aligned}$$

Induced relation?

$$R_{\boxslash} \xleftarrow{\;\color{red}{?}\;} (\mathcal{O}X, \triangle, \triangledown)$$

Induced relation?

$$R_{\triangle\!\!\!\!\nabla} \xleftarrow{\quad ?\quad} \hspace{-0.5em}\dashv (\mathcal{O}X, \triangle, \nabla)$$

Idea: generate a congruence $\langle \sim \rangle$ on the frame $\mathcal{O}X \otimes \mathcal{O}X$

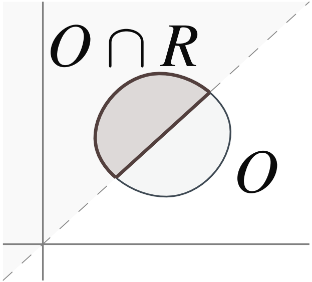
Induced relation?

$$R_{\triangle\!\!\!\nabla} \xleftarrow{\;\;?\;\;} (\mathcal{O}X, \triangle, \nabla)$$

Idea: generate a congruence $\langle\sim\rangle$ on the frame $\mathcal{O}X \otimes \mathcal{O}X$

by $\sim$ defined on rectangles $U \otimes V$ in terms of $\triangle$, ∇

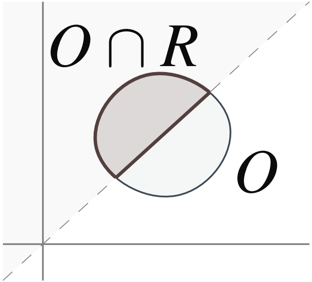
Spatial intuition



$$r \colon R \hookrightarrow S \times S$$

$$r^{-1}(O) = O \cap R$$

Spatial intuition

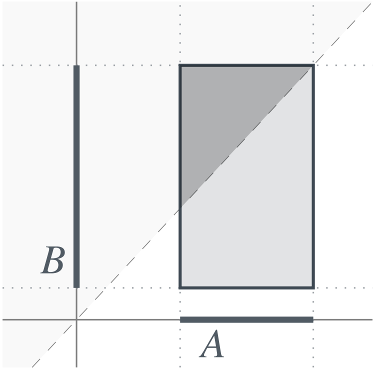


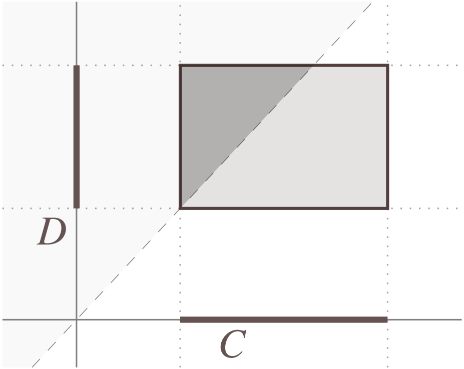
$$r\colon R \hookrightarrow S \times S$$

$$r^{-1}(O) = O \cap R$$

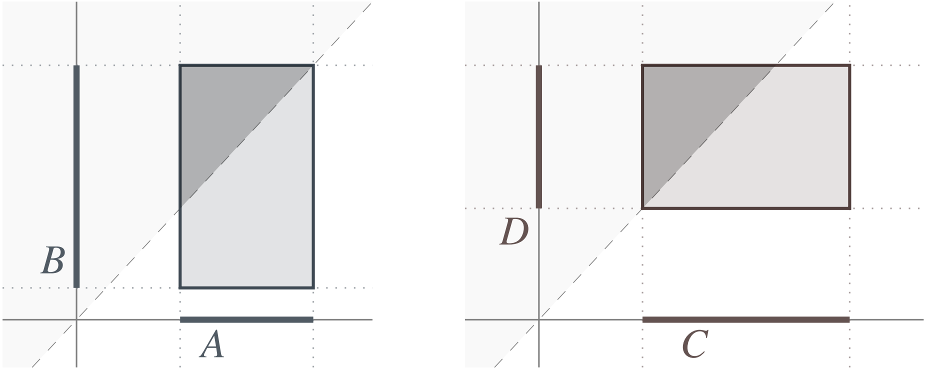
$$O \equiv O' \quad \iff \quad O \cap R = O' \cap R$$

Congruence on rectangles



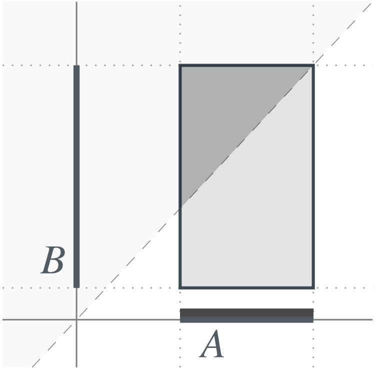


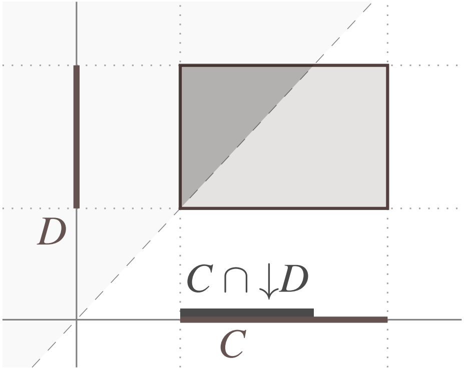
Congruence on rectangles



$$(A \times B) \cap R = (C \times D) \cap R \quad \Longleftrightarrow \quad \textbf{?}$$

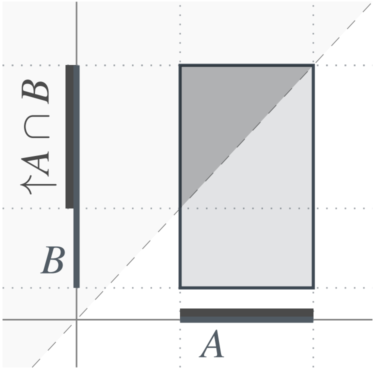
Congruence on rectangles

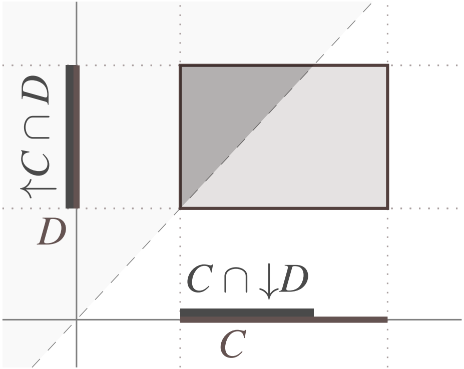




$$(A \times B) \cap R = (C \times D) \cap R \quad \Longleftrightarrow \quad \textbf{?}$$

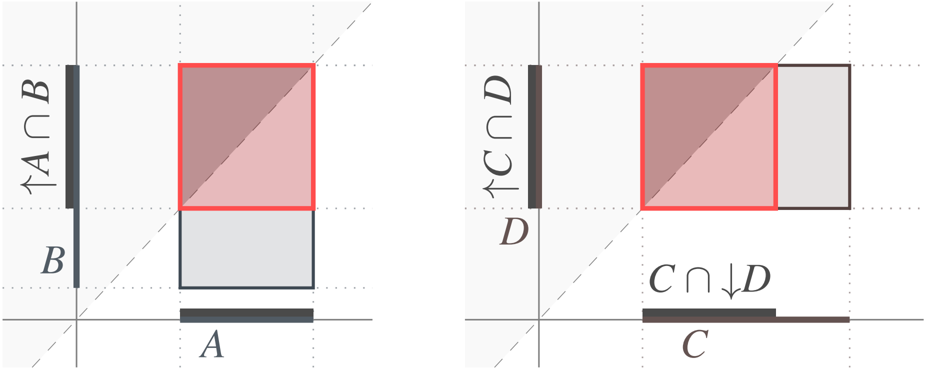
Congruence on rectangles





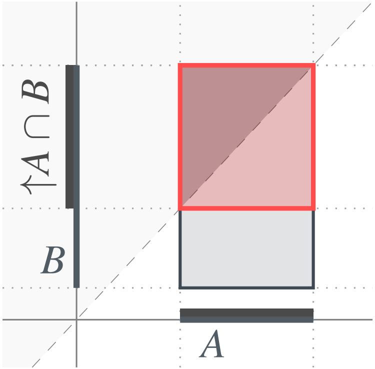
$$(A \times B) \cap R = (C \times D) \cap R \quad \Longleftrightarrow \quad ?$$

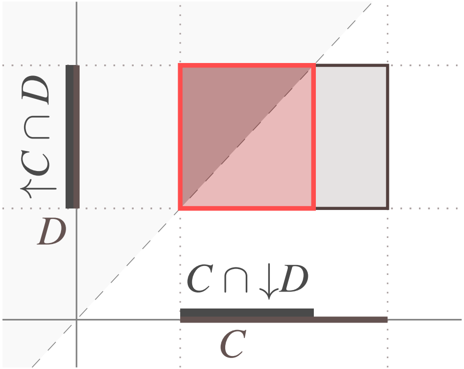
Congruence on rectangles



$$(A \times B) \cap R = (C \times D) \cap R \iff ?$$

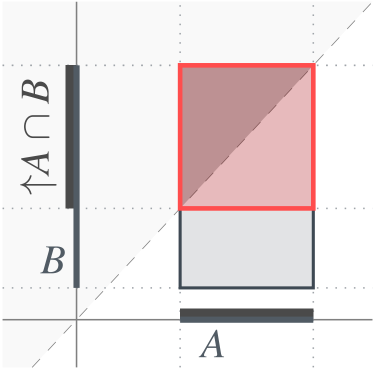
Congruence on rectangles

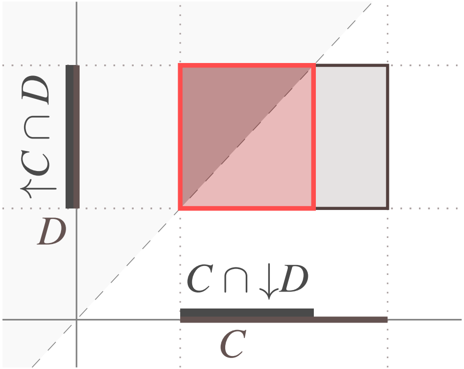




$$(A \times B) \cap R = (C \times D) \cap R \iff \boxed{\begin{array}{l} A \cap \downarrow B = C \cap \downarrow D \\ \uparrow A \cap B = \uparrow C \cap D \end{array}}$$

Congruence on rectangles





$$A \otimes B \sim C \otimes D \iff \boxed{\begin{array}{c} A \wedge \nabla B = C \wedge \nabla D \\ \Delta A \wedge B = \Delta C \wedge D \end{array}}$$

Induced relation

[MPP17] M. A. Moshier et al., *"Generating Sublocales by Subsets and Relations"*, 2017.

Definition. Given $(\mathcal{O}X, \triangle, \triangledown)$, construct $R_{\triangle\!\!\!\triangledown}$ via:

$$\mathcal{O}R_{\triangle\!\!\!\triangledown} := \mathcal{O}X \otimes \mathcal{O}X / \langle \sim \rangle.$$

Induced relation [MPP17]

[MPP17] M. A. Moshier et al., *"Generating Sublocales by Subsets and Relations"*, 2017.

Definition. Given $(\mathcal{O}X, \triangle, \nabla)$, construct $R_{\triangle\!\!\!\!\nabla}$ via:

$$\mathcal{O}R_{\triangle\!\!\!\!\nabla} := \mathcal{O}X \otimes \mathcal{O}X / \langle \sim \rangle.$$

Theorem. $R_{\triangle\!\!\!\!\nabla}$ is an oc localic relation

Induced relation [MPP17]

[MPP17] M. A. Moshier et al., *“Generating Sublocales by Subsets and Relations”*, 2017.

Definition. Given $(\mathcal{O}X, \vartriangle, \triangledown)$, construct $R_{\vartriangle\!\!\triangledown}$ via:

$$\mathcal{O}R_{\vartriangle\!\!\triangledown} := \mathcal{O}X \otimes \mathcal{O}X / \langle\sim\rangle.$$

Theorem. $R_{\vartriangle\!\!\triangledown}$ is an oc localic relation with

$$\Uparrow\, = \vartriangle \quad \text{and} \quad \Downarrow\, = \triangledown\,.$$

Induced relation

$$
\begin{aligned}
\mathsf{rel}\colon \boxbslash\mathbf{Frm}^{\mathrm{op}} &\longrightarrow \mathbf{rocLoc} \\
(\mathcal{O}X, \vartriangle, \triangledown) &\longmapsto (X, R_{\boxbslash}) \\
f^{-1} &\longmapsto f
\end{aligned}
$$

The adjunction

Theorem. There is an adjunction

$$\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left=2, "\mathsf{cone}"] \arrow[r, phantom, "\perp"] & \boxbslash\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left=2, "\mathsf{rel}"] \end{tikzcd}$$

The adjunction

Theorem. There is an adjunction

$$\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left, "\mathsf{cone}"] \arrow[r, phantom, "\perp"] & \boxslash\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{rel}"] \end{tikzcd}$$

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The adjunction

Theorem. There is an adjunction

$$\mathbf{rocLoc} \underset{\mathsf{rel}}{\overset{\mathsf{cone}}{\underset{\perp}{\rightleftarrows}}} \boxslash\mathbf{Frm}^{\mathrm{op}}$$

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta\colon (X, R) \to (X, R_{\Uparrow\downarrow})$
is an inclusion $R \subseteq R_{\Uparrow\downarrow}$

The adjunction

Theorem. There is an adjunction

$$\begin{tikzcd} \mathbf{rocLoc} \arrow[r, shift left=2, "\mathsf{cone}"] \arrow[r, phantom, "\perp"] & \triangle\!\!\!\nabla\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left=2, "\mathsf{rel}"] \end{tikzcd}$$

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta \colon (X, R) \to (X, R_{\uparrow\downarrow})$
is an inclusion $R \subseteq R_{\uparrow\downarrow}$

Corollary. $R \subseteq R_{\triangle\!\!\!\nabla}$ iff $\uparrow \leqslant \triangle$ and $\downarrow \leqslant \nabla$

Fixed points

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta\colon (X, R) \to (X, R_{\Uparrow\!\Downarrow})$
is an inclusion $R \subseteq R_{\Uparrow\!\Downarrow}$

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

Fixed points

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta\colon (X, R) \to (X, R_{\Uparrow\downarrow})$
is an inclusion $R \subseteq R_{\Uparrow\downarrow}$

Proposition. The inclusion $R \subseteq R_{\Uparrow\downarrow}$ is *(strongly) dense*.

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

Fixed points

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta \colon (X, R) \to (X, R_{\Uparrow\downarrow})$
is an inclusion $R \subseteq R_{\Uparrow\downarrow}$

Proposition. The inclusion $R \subseteq R_{\Uparrow\downarrow}$ is *(strongly) dense*.

Corollary. If oc R is *(weakly) closed* then $R = R_{\Uparrow\downarrow}$.

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

Fixed points

The counit is $\varepsilon = \mathrm{id}$, since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta\colon (X, R) \to (X, R_{\Uparrow\!\Downarrow})$ is an inclusion $R \subseteq R_{\Uparrow\!\Downarrow}$

Proposition. The inclusion $R \subseteq R_{\Uparrow\!\Downarrow}$ is *(strongly) dense*.

Corollary. If oc R is *(weakly) closed* then $R = R_{\Uparrow\!\Downarrow}$.

e.g. if $q\colon X \to Q$ is open, its *kernel pair* $X \times_Q X$ is fixed

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

Fixed points

The counit is $\varepsilon = \mathrm{id}$,
since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta \colon (X, R) \to (X, R_{\Uparrow\Downarrow})$
is an inclusion $R \subseteq R_{\Uparrow\Downarrow}$

Proposition. The inclusion $R \subseteq R_{\Uparrow\Downarrow}$ is *(strongly) dense*.

Corollary. If oc R is *(weakly) closed* then $R = R_{\Uparrow\Downarrow}$.

e.g. if $q \colon X \to Q$ is open, its *kernel pair* $X \times_Q X$ is fixed
e.g. if R is an oc equivalence relation, $R_{\Uparrow\Downarrow} \cong X \times_{X/R} X$

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

Fixed points

The counit is $\varepsilon = \mathrm{id}$, since $\mathsf{cone} \circ \mathsf{rel} = \mathrm{id}$

The unit $\eta\colon (X, R) \to (X, R_{\Uparrow\downarrow})$ is an inclusion $R \subseteq R_{\Uparrow\downarrow}$

Proposition. The inclusion $R \subseteq R_{\Uparrow\downarrow}$ is *(strongly) dense*.

Corollary. If oc R is *(weakly) closed* then $R = R_{\Uparrow\downarrow}$.

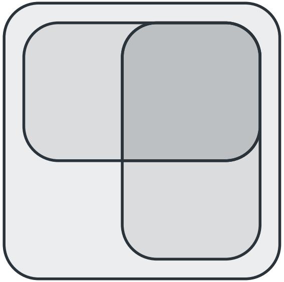
e.g. if $q\colon X \to Q$ is open, its *kernel pair* $X \times_Q X$ is fixed
e.g. if R is an oc equivalence relation, $R_{\Uparrow\downarrow} \cong X \times_{X/R} X$

Corollary ([Koc89]). Any weakly closed oc equivalence relation is the kernel pair of its quotient map $X \twoheadrightarrow X/R$.

[Koc89] A. Kock, *"A Godement Theorem for Locales"*, 1989.

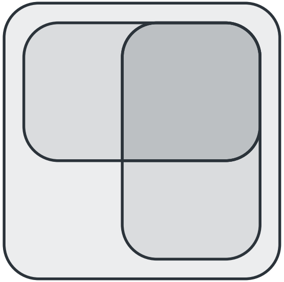
A small counterexample

$$\mathcal{O}\mathbb{S} = \{\bot < a < \top\}$$



A small counterexample

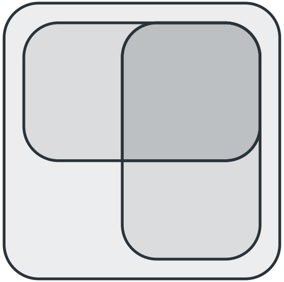
$\mathcal{O}\mathbb{S} = \{\bot < a < \top\}$



Take R open sublocale
$(a \otimes \top) \vee (\top \otimes a)$

A small counterexample

$$\mathcal{O}\mathbb{S} = \{\bot < a < \top\}$$

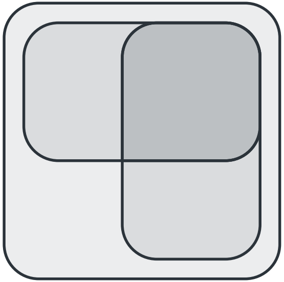


Take R open sublocale $(a \otimes \top) \vee (\top \otimes a)$

Cones $\Uparrow = \Downarrow$ send $\bot \mapsto \bot$ and $a, \top \mapsto \top$

A small counterexample

$\mathcal{O}\mathbb{S} = \{\bot < a < \top\}$



Take R open sublocale $(a \otimes \top) \vee (\top \otimes a)$

Cones $\Uparrow = \Downarrow$ send $\bot \mapsto \bot$ and $a, \top \mapsto \top$

$R_{\Uparrow\Downarrow} = \mathbb{S} \times \mathbb{S} \supsetneq R$

Properties in terms of cones

Proposition. If an oc R is

- reflexive then $\mathrm{id} \leqslant \Uparrow$ and $\mathrm{id} \leqslant \Downarrow$
- transitive then $\Uparrow^2 \leqslant \Uparrow$ and $\Downarrow^2 \leqslant \Downarrow$
- symmetric then $\Uparrow = \Downarrow$

Properties in terms of cones

Proposition. an oc $R = R_{\Uparrow\!\Downarrow}$ is

- reflexive *iff* $\mathrm{id} \leqslant \Uparrow$ and $\mathrm{id} \leqslant \Downarrow$
- transitive *iff* $\Uparrow^2 \leqslant \Uparrow$ and $\Downarrow^2 \leqslant \Downarrow$
- symmetric *iff* $\Uparrow = \Downarrow$

Localic Esakia duality

Localic Esakia duality

$$\mathbf{CohFrm} \overset{\text{[Tow97]}}{\longleftrightarrow} \mathbf{PLoc}$$

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$\mathbf{CohFrm} \overset{\text{[Tow97]}}{\leftsquigarrow\!\rightsquigarrow} \mathbf{PLoc}$$

[Tow97] C. Townsend, *“Localic Priestley Duality”*, 1997.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$\begin{array}{ccc} \mathbf{CohFrm} & \overset{\text{[Tow97]}}{\leftrightsquigarrow} & \mathbf{PLoc} \\ \cup\!\mid & & \\ \mathbf{HFrm} & & \end{array}$$

Heyting frames
[BCM23]

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.
[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$\begin{tikzcd} \mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "\text{[Tow97]}"] & & \mathbf{PLoc} \\ \mathbf{HFrm} \arrow[u, phantom, "\cup\!\mid" rotate=180] & & \mathbf{ELoc} \arrow[u, phantom, "\cup\!\mid" rotate=180] \end{tikzcd}$$

Heyting frames
[BCM23]

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.
[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$\begin{array}{ccc} \mathbf{CohFrm} & \xleftrightarrow{\text{[Tow97]}} & \mathbf{PLoc} \\ \cup\!| & & \cup\!| \\ \mathbf{HFrm} & & \mathbf{ELoc} \subseteq \mathbf{rosLoc} \end{array}$$

Heyting frames
[BCM23]

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.

[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

Heyting frames
[BCM23]

$$\begin{array}{ccccc}
\mathbf{CohFrm} & \overset{\text{[Tow97]}}{\leftrightsquigarrow} & \mathbf{PLoc} & & \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \\
\mathbf{HFrm} & \overset{\color{red}?}{\leftrightsquigarrow} & \mathbf{ELoc} & \subseteq & \mathbf{rosLoc}
\end{array}$$

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.

[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

Heyting frames
[BCM23]

$$
\begin{tikzcd}
\mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "\text{[Tow97]}"] & & \mathbf{PLoc} & & \\
\mathbf{HFrm} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, leftrightsquigarrow, "\textcolor{red}{?}"] & & \mathbf{ELoc} \arrow[u, phantom, sloped, "\subseteq"] & \subseteq & \mathbf{rosLoc} \arrow[dll, "\mathsf{cone}"', bend right=10] \\
& & \nabla\mathbf{Frm} \arrow[urr, "\mathsf{rel}"', bend right=10] \arrow[urr, phantom, "\perp"] & &
\end{tikzcd}
$$

[Tow97] C. Townsend, *“Localic Priestley Duality”*, 1997.
[BCM23] G. Bezhanishvili et al., *“A Frame-Theoretic Perspective on Esakia Duality”*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

Heying frames
[BCM23]

$$
\begin{tikzcd}
\mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "{[\text{Tow97}]}"] \arrow[d, phantom, "\cup\!|"] & & \mathbf{PLoc} \arrow[d, phantom, "\cup\!|"] & \\
\mathbf{HFrm} \arrow[rr, leftrightsquigarrow, "\color{red}?"] & & \mathbf{ELoc} \arrow[r, phantom, "\subseteq"] \arrow[dl, leftrightsquigarrow] & \mathbf{rosLoc} \arrow[dl, bend right, "\mathsf{cone}"'] \\
& \mathbf{EFrm} \arrow[r, phantom, "\subseteq"] & \nabla\mathbf{Frm} \arrow[ur, bend right, "\mathsf{rel}"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.

[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$
\begin{tikzcd}
\mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "{[\text{Tow97}]}"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & & \mathbf{PLoc} \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & \\
\mathbf{HFrm} \arrow[rr, leftrightsquigarrow, "\textcolor{red}{\mathbf{?}}"] & & \mathbf{ELoc} \arrow[r, phantom, "\subseteq"] & \mathbf{rosLoc} \arrow[dl, shift right, "\mathsf{cone}"'] \arrow[dl, phantom, "\perp"] \\
& \mathbf{EFrm} \arrow[ur, leftrightsquigarrow] \arrow[r, phantom, "\subseteq"] & \nabla\mathbf{Frm} \arrow[ur, shift right, "\mathsf{rel}"'] &
\end{tikzcd}
$$

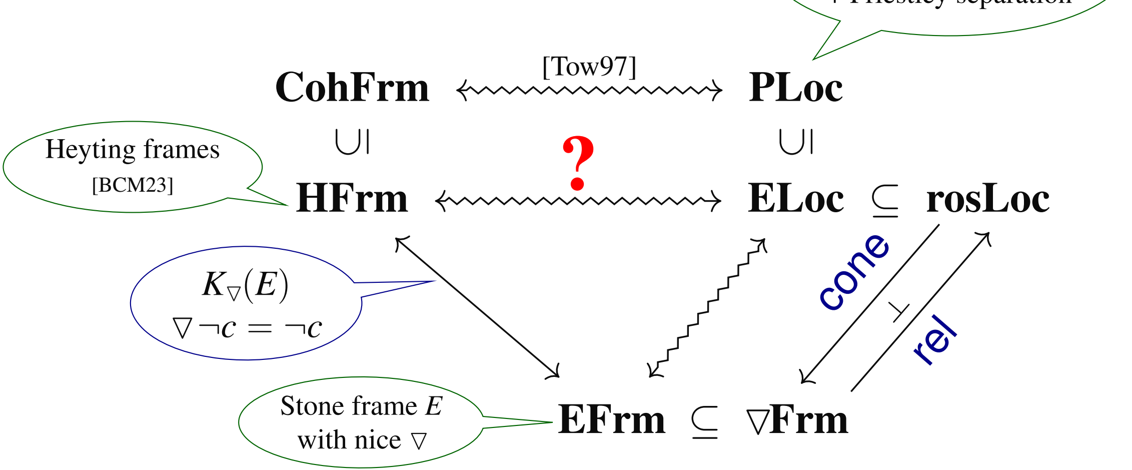
Heying frames
[BCM23]

Stone frame E
with nice ∇

[Tow97] C. Townsend, *“Localic Priestley Duality”*, 1997.

[BCM23] G. Bezhanishvili et al., *“A Frame-Theoretic Perspective on Esakia Duality”*, 2023.

Localic Esakia duality



[Tow97] C. Townsend, *“Localic Priestley Duality”*, 1997.

[BCM23] G. Bezhanishvili et al., *“A Frame-Theoretic Perspective on Esakia Duality”*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

$$
\begin{tikzcd}
\mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "{[\text{Tow97}]}"] & & \mathbf{PLoc} & & \\
\mathbf{HFrm} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, leftrightsquigarrow, "\checkmark"] \arrow[dr, leftrightarrow] & & \mathbf{ELoc} \arrow[u, phantom, sloped, "\subseteq"] \arrow[dl, leftrightsquigarrow] & \subseteq & \mathbf{rosLoc} \arrow[dl, bend right, "\mathsf{cone}"'] \arrow[dl, phantom, "\dashv"] \\
& \mathbf{EFrm} & \subseteq & \nabla\mathbf{Frm} \arrow[ur, bend right, "\mathsf{rel}"'] &
\end{tikzcd}
$$

Heyting frames
[BCM23]

$K_\nabla(E)$
$\nabla\neg c = \neg c$

Stone frame E
with nice ∇

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.
[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Localic Esakia duality

Stone locales X
+ closed localic $\leqslant$
+ Priestley separation

Heyting frames
[BCM23]

$K_\triangledown(E)$
$\triangledown\neg c = \neg c$

Stone frame E
with nice $\triangledown$

$$\begin{tikzcd}
\mathbf{CohFrm} \arrow[rr, leftrightsquigarrow, "{[\text{Tow97}]}"] \arrow[d, phantom, "\cup\!|"] & & \mathbf{PLoc} \arrow[d, phantom, "\cup\!|"] & & \\
\mathbf{HFrm} \arrow[rr, leftrightsquigarrow, "\checkmark"] \arrow[dr, leftrightarrow] & & \mathbf{ELoc} \arrow[dl, leftrightsquigarrow] & \subseteq & \mathbf{rosLoc} \arrow[dl, shift right=2, "\text{cone}"'] \arrow[dl, phantom, "\perp"] \\
& \mathbf{EFrm} & \subseteq & \triangledown\mathbf{Frm} \arrow[ur, shift right=2, "\text{rel}"'] &
\end{tikzcd}$$

Preprint: coming soon!

[Tow97] C. Townsend, *"Localic Priestley Duality"*, 1997.
[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Future work

- Recover $R \rightarrowtail X \times X$ from more general $\uparrow, \downarrow\colon \mathrm{S}\ell(X) \longrightarrow \mathrm{S}\ell(X)$? (there: $\uparrow A = t\left[s^{-1}[A]\right]$ or $R \circ A$)

[Gou26] J. Goubault-Larrecq, *Stone Duality for Preordered Topological Spaces*, 2026.
[JT51] B. Jónsson et al., *"Boolean Algebras with Operators. Part I"*, 1951.

Future work

- Recover $R \rightarrowtail X \times X$ from more general $\uparrow, \downarrow\colon \mathrm{S}\ell(X) \longrightarrow \mathrm{S}\ell(X)$? (there: $\uparrow A = t\left[s^{-1}[A]\right]$ or $R \circ A$)
- Relations to biframes, d-frames, ad-frames [Gou26]?
- Relations to modal logics?
- Relations to BAOs [JT51]?

[Gou26] J. Goubault-Larrecq, *Stone Duality for Preordered Topological Spaces*, 2026.
[JT51] B. Jónsson et al., *"Boolean Algebras with Operators. Part I"*, 1951.

Future work

- Recover $R \rightarrowtail X \times X$ from more general $\uparrow, \downarrow\colon \mathrm{S}\ell(X) \longrightarrow \mathrm{S}\ell(X)$? (there: $\uparrow A = t\left[s^{-1}[A]\right]$ or $R \circ A$)
- Relations to biframes, d-frames, ad-frames [Gou26]?
- Relations to modal logics?
- Relations to BAOs [JT51]?

nvds.site

arXiv:2605.04038

Thanks!

[Gou26] J. Goubault-Larrecq, *Stone Duality for Preordered Topological Spaces*, 2026.

[JT51] B. Jónsson et al., *“Boolean Algebras with Operators. Part I”*, 1951.