A Generic Construction of Free Algebras in Varieties of Hilbert Algebras and Brouwerian Semilattices

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TACL2026

27–31 July 2026

From Heyting algebras to Hilbert algebras

1. Brouwerian Semilattices ($\mathcal{B}r$) are $\{\wedge, \to, 1\}$-subreducts of Heyting algebras.
2. Hilbert algebras ($\mathcal{H}i$) are $\{\to, 1\}$-subreducts of Brouwerian semilattices (hence, of Heyting algebras).
3. All of these are CD, have EDPC, have CEP and are Fregean (that is, congruence 1-orderable and 1-regular).
4. $\mathcal{B}r$ and $\mathcal{H}i$ are locally finite.
5. $\mathcal{H}i$ is congruence 3-permutable, but not congruence permutable.
6. For every finite $\mathbf{A} \in \mathcal{H}i$, there exists a unique free extension $\langle \mathbf{A} \rangle \in \mathcal{B}r$ (Celani, Jansana).
7. The lattice of subvarieties of $\mathcal{B}r$ embeds into the lattice of subvarieties of $\mathcal{H}i$.

SI algebras: the masting construction

For any $\mathbf{A} \in \mathcal{H}i$ we write $\mathbf{A}^{\oplus}$ for the ordinal sum $\mathbf{A} \oplus \mathbf{2}$.
Intuitively, $\mathbf{A}^{\oplus}$ arises by putting a 'mast' on top of $\mathbf{A}$.

Theorem

Let $\mathcal{V} \subseteq \mathcal{H}i$ or $\subseteq \mathcal{B}r$, and let $A \in \mathcal{V}$. Then

1. *$A^{\oplus}$ is subdirectly irreducible.*
2. *Every subdirectly irreducible $\mathcal{V}$-algebra is of that form for some $A \in \mathcal{V}$ (so it has a unique coatom, denoted $\bigstar$).*

Moreover, $A^{\oplus} \smallsetminus \{\bigstar\}$ is a subuniverse of $A^{\oplus}$.

- 'Moreover' is responsible for local finiteness.
- Heyting algebras satisfy (1) and (2), but not 'moreover'.

Irreducible elements in finite algebras

Definition

For any $\mathbf{A} \in \mathcal{H}i$, define the set $I(A)$ of irreducible elements, putting:

$$I(A) = \{a \in A : \theta(1, a) \in \mathcal{J}(\mathbf{Con}\,\mathbf{A})\}$$

where $\mathcal{J}(\mathbf{Con}\,\mathbf{A})$ is the poset of join-irreducible congruences of $\mathbf{Con}\,\mathbf{A}$ ordered by inclusion.

A purely lattice-theoretic fact:

- If A is finite, then $\mathcal{J}(\mathbf{Con}\,\mathbf{A}) \cong \mathbf{Cm}\,\mathbf{A}$, where $\mathbf{Cm}\,\mathbf{A}$ is the lattice of completely meet-irreducible congruences of $\mathbf{A}$.

Characterising $I(A)$

Theorem

Let $\mathbf{A} \in \mathcal{H}i$ be finite. Then for any $a \in A \setminus \{1\}$, the following are equivalent:

- $a \in I(A)$
- $\theta(1, a) \in \mathcal{J}(\mathbf{Con\,A})$
- $\forall m \in A : m \to a \in \{1, a\}$

If $\mathbf{A} \in \mathcal{B}r$, then $a \in A$ is irreducible if and only if a is meet-irreducible.

- We can easily manufacture algebras with only irreducible elements: in fact, $I(A)$ is always a subuniverse of $\mathbf{A}$.

Algebras of antichains

It is well known how to make algebras out of upsets. We will make algebras out of antichains.

Definition

For any poset P, let $Ac(P)$ be the set of all its antichains. Put:

- a binary relation $\leq$ on $Ac(P)$ putting $C \leq B$ if $B \subseteq \downarrow C$.
- a binary operation $\to$ on $Ac(P)$ putting $C \to B = \{b \in B : \uparrow b \cap C = \emptyset\}$
- a binary operation $\wedge$ on $Ac(P)$ putting $B \wedge C = \max(B \cup C)$.

Definition

For any poset P we define the algebras:

- $\mathbf{Ac}(P) = (Ac(P); \to, \emptyset)$
- $\mathbf{Acm}(P) = (Ac(P); \wedge, \to, \emptyset)$

Antichain selection function

- For any algebra $\mathbf{Ac}(P)$, define a map $\varphi : Ac(P) \to Up(P)$, putting $\varphi(X) = P \setminus {\downarrow}X$ for any antichain X.
- Then φ is an embedding of $(Ac(P), \leq)$ into $(Up(P); \subseteq)$. If P is finite, then φ is an isomorphism of posets.
- Instead of the usual representation by upsets we want to work with a representation by antichains. The next definition prepares the ground.

Definition

For any $\mathbf{A} \in \mathcal{H}i$ (or $\mathcal{B}r$), let $S\colon A \to Ac(\mathrm{Cm}\,\mathbf{A})$ be defined by:

$$S(a) = \{\mu \in \mathrm{Cm}\,\mathbf{A} : \mu \vee \theta(1, a) = \mu^+\}$$

where μ^+ is the unique cover of the completely meet-irreducible μ.

The representation

- Note that for any $\mathbf{A}$, any $a \in A$, and any $\mu \in \operatorname{Cm}\mathbf{A}$ we have $a/\mu = \bigstar^{\mathbf{A}/\mu} \iff \mu \vee \theta(a,1) = \mu^+$.
- Hence $S(a) = \{\mu \in \operatorname{Cm}\mathbf{A} : \mu \vee \theta(1,a) = \mu^+\}$ sends a to a special antichain in $\mathbf{Cm}\,\mathbf{A}$, namely the antichain of these μ that 'star' a in the quotient.

Theorem (Representation)

Let $\mathbf{A} \in \{\mathcal{H}i, \mathcal{B}r\}$. The map S is a monomorphism, and:

1. *If $\mathbf{A} \in \mathcal{H}i$, then $\mathbf{A} \cong S(A) \leq \mathbf{Ac}(\mathbf{Cm}\,\mathbf{A})$.*
2. *If $\mathbf{A} \in \mathcal{B}r$, then $\mathbf{A} \cong S(A) \leq \mathbf{Acm}(\mathbf{Cm}\,\mathbf{A})$.*
3. *If $\mathbf{A} \in \mathcal{B}r$ is finite, then $\mathbf{A} \cong S(A) \cong \mathbf{Acm}(\mathbf{Cm}\,\mathbf{A})$.*

An analogue of (3) for Hilbert algebras does not hold.

Representation, irreducibles, congruences

Theorem

Let $\mathbf{A} \in \mathcal{H}i$. *If* $\mathbf{A}$ *is finite, then:*

1. $S(I(\mathbf{A})) = \{\{\mu\} : \mu \in \mathrm{Cm}\,\mathbf{A}\}$.
2. $\mathbf{Con}\,\mathbf{Ac}(\mathrm{Cm}\,\mathbf{A}) = \mathbf{Con}\,\mathbf{Acm}(\mathrm{Cm}\,\mathbf{A}) \cong \mathbf{Con}\,\mathbf{A}$.

Theorem (The All Subsets Principle)

Let $\mathbf{A} \in \mathcal{H}i$. *If* $\mathbf{A}$ *is finite, then for any* $a \in A$ *and any* $L \subseteq \mathrm{Cm}\,\mathbf{A}$ *we have*

$$L \subseteq S(a) \Rightarrow L \in S(A).$$

Any subset of an S–image of something must be an S–image of something itself.

Free algebra reassembly idea

- Let $\mathcal{V}$ be a variety of Hilbert algebras, so n generated free algebras $\mathsf{F}_{\mathcal{V}}(n)$ exist, and are finite.
- Otherwise they are black boxes.
- We will dismantle them, and then reassemble.

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- The poset $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)$ is composed of layers.
- The first layer $P_1(n)$ corresponds to 2-element quotients.
- The layer $P_{k+1}(n)$ corresponds to quotients arising by 'masting' subproducts of $P_k(n)$-quotients.

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- To stay within $\mathcal{V}$ we need to know which (subproducts of) algebras from $\mathcal{V}$ can be 'masted' in $\mathcal{V}$.

Layer $P_1(n)$

Let $X = \{x_1, \ldots, x_n\}$ be the set of free generators of $\mathsf{F}_{\mathcal{V}}(n)$.

Lemma

For any $J \subset X$ let $f_J \colon X \to \{0, 1\}$ be the map given by:

$$f_J(x) = \begin{cases} 1 & \textit{if } x \in J, \\ 0 & \textit{if } x \notin J, \end{cases}$$

where $\{0, 1\}$ is the universe of $\mathbf{2}$. Let $\overline{f}_J \colon \mathsf{F}_{\mathcal{V}}(n) \to \mathbf{2}$ be the homomorphism uniquely extending f_J, and let $\mu_J = \ker \overline{f}_J$. Then

$$P_1(n) = \{\mu_J : J \subset X\}.$$

The reason for J being a proper subset of X is that at least one variable must be sent to $\bigstar$, which in this case is 0.

Higher layers

Lemma

Let $\mathcal{V} \subseteq \mathsf{Hi}$ and $k \geq 2$. For any $G \in \mathrm{Up}\left(\bigcup_{r=1}^{k-1} P_r(n)\right)$ such that $G \cap P_{k-1}(n) \neq \emptyset$ and $\left(\mathbf{F}_{\mathcal{V}}(n)/\bigcap G\right)^{\oplus} \in \mathcal{V}$, and for any L such that $\emptyset \subseteq L \subset X \cap 1/\bigcap G$, let $f_L^G \colon X \to \left(\mathbf{F}_{\mathcal{V}}(n)/\bigcap G\right)^{\oplus}$ be the map given by

$$f_L^G(x) = \begin{cases} 1 & \text{if } x \in L, \\ \bigstar & \text{if } x \in (1/\bigcap G) \smallsetminus L, \\ x/\bigcap G & \text{if } x \notin 1/\bigcap G. \end{cases}$$

Let $\overline{f}_L^G$ be the homomorphism uniquely extending f_L^G and let $\mu_L^G = \ker \overline{f}_L^G$. Then

$$P_k(n) = \left\{ \mu_L^G : G \in \mathrm{Up}\left(\bigcup_{r=1}^{k-1} P_r(n)\right),\ G \cap P_{k-1}(n) \neq \emptyset,\ L \subset X \cap 1/\bigcap G \right\}.$$

Furthermore, let $m = \min\{n, \ell\}$, where ℓ is the length of the longest chain in $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)$. Then, for $r > m$ we have $P_r(n) = \emptyset$; hence $\bigcup_{k=1}^{\infty} P_k(n) = \bigcup_{k=1}^{m} P_k(n)$.

Legend:

- G – some selection of already considered quotients.
- $G \cap P_{k-1}(n) \neq \emptyset$ – must contain at least one from immediately preceding stage.
- $\mathbf{F}_{\mathcal{V}}(n)/\bigcap G$ – quotient out the subdirect product.
- $\left(\mathbf{F}_{\mathcal{V}}(n)/\bigcap G\right)^{\oplus} \in \mathcal{V}$ – check if the masted quotient is still in $\mathcal{V}$.
- $f_L^G(x)$ – where to send generators.

Algorithm for $\mathbf{Cm}\,\mathsf{F}_{\mathsf{Hi}}(n)$

Require: $J \subseteq X$, $1 \leq i \leq n$ ▷ Points are pairs (J, i)

For s set U of points, $\textsc{Proj}(U) := \{J : (J, i) \in U \text{ for some } i\}$

Let $P_1(n) = \{(J, 1) : J \subsetneq X\}$; $\leq \leftarrow =$

function $\textsc{BuildNextLayer}(\bigcup_{i=1}^{k-1} P_i(n))$

  $P_k(n) \leftarrow \emptyset$

  for $G \in \mathrm{Up}\left(\bigcup_{r=1}^{k-1} P_r(n)\right)$, $G \cap P_{k-1}(n) \neq \emptyset$, $L \subsetneq \bigcap \textsc{Proj}(G)$ **do**

    $P_k(n) \leftarrow P_k(n) \cup \{(L, k)\}$

    for $(J, j) \in G$ **do**

      $\leq \leftarrow \leq \cup \langle (L, k), (J, j) \rangle$

    end for

  end for

end function

for $k \in \{2, \ldots, n\}$ **do**

  $\textsc{BuildNextLayer}(\bigcup_{i=1}^{k-1} P_i(n))$

end for

Free algebras in $\mathcal{H}i$

Recall the map $S\colon \mathbf{A} \to \mathbf{Ac}(\mathbf{Cm}\,\mathbf{A})$. Applying it to $\mathbf{F}_{\mathcal{V}}(n)$, we get

$$S(x) = \{\mu \in \mathrm{Cm}\,\mathbf{F}_{\mathcal{V}}(n) : \mu \vee \theta(1, x) = \mu^+\},$$

for any $x \in X$. So S selects certain antichains from $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)$.

Theorem (Free Hilbert algebras)

Let $\mathcal{V} \subseteq \mathcal{H}i$. For any $n > 0$, we have

$$\mathbf{F}_{\mathcal{V}}(n) \cong \left(\bigcup_{i=1}^{n} \mathcal{P}(S(x_i)); \to, \emptyset \right) \leq \mathbf{Ac}(\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)).$$

For example $|\mathbf{F}_{\mathcal{H}i}(3)| = 25\,165\,802$.

Free algebras in $\mathcal{B}r$

- Let $f\colon \Lambda(\mathcal{B}r) \to \Lambda(\mathcal{H}i)$ be the embedding mentioned earlier.
- We will write $\widehat{\mathcal{V}}$ for $f^{-1}(\mathcal{V})$. For every $\mathcal{W} \in \Lambda(\mathcal{B}r)$ we have $\mathcal{W} = \widehat{\mathcal{V}}$ for a unique $\mathcal{V} \in \Lambda(\mathcal{H}i)$.

Theorem (Free Brouwerian semilattices)

Let $\widehat{\mathcal{V}} \subseteq \mathcal{B}r$. For any $n > 0$, we have

$$\mathbf{F}_{\widehat{\mathcal{V}}}(n) \cong \left\langle \bigcup_{i=1}^{n} \mathcal{P}(S(x_i)); \to, \wedge, \emptyset \right\rangle \cong \mathbf{Acm}(\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)),$$

where the angle brackets denote the free extension of $\mathbf{F}_{\mathcal{V}}(n)$.

What was tried before was the opposite: construct $\mathbf{F}_{\mathcal{B}r}(n)$ first and then take only these elements that you can generate by $\to$ only.

Other free algebras we can (effectively) do

1. Varieties of Hilbert algebras with zero and Brouwerian semilattices with zero.
2. Varieties of bounded height, and bounded width (with or without zero).
3. Free spectrum for the linear case:

Theorem

For every $n \geq 1$,

$$|\mathbf{F}_{\mathrm{Hb}_1}(n)| = (-1)^{n-1}\left[2 + \frac{1}{2}\sum_{m=1}^{n-1}(-1)^m \binom{n}{m} 16^{a(m)}\right].$$

where $a(n)$ is the n-th Fubini number, that is, the number of surjections from a fixed n to any m.

Structural completeness

- $\mathcal{H}i$ (Prucnal), $\mathcal{B}r$ and $\mathcal{B}o$ (Minari, Wroński) are hereditarily structurally complete, whereas $\mathcal{H}o$ (Wroński) is not.

Theorem

Let $\mathcal{V} \subseteq \mathcal{H}o$. If every finite subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ has a set of generators $\{a_1, \ldots, a_n\}$, where $a_1 = \bigstar$, such that $\mathbf{A} \models a_2 \to (a_3 \to \ldots \to (a_n \to 0) \ldots) \neq 1$, then $\mathcal{V}$ is hereditarily structurally complete.

Theorem

Let $\mathcal{V} \subseteq \mathcal{H}o$ and let $t_1, \ldots, t_k, r \in \mathrm{Term}_{\{\to,0\}}(n)$. If for every $i \in \{1, \ldots, k\}$ we have $t_i(\overline{1}) = 1$, then the following are equivalent:

1. $\mathbf{F}_{\mathcal{V}}(n) \models \bigwedge_{i=1}^{k} t_i = 1 \implies r = 1$,
2. $\mathbf{F}_{\mathcal{V}}(n) \models t_1 \to (t_2 \to \ldots \to (t_k \to r) \ldots) = 1$.

G. Bezhanishvili, N. Bezhanishvili, L. Carai, D. Gabelaia, S. Ghilardi, and M. Jibladze.
Diego's theorem for nuclear implicative semilattices.
Indag. Math. (N.S.), 32(2):498–535, 2021.

S. A. Celani and L. Cabrer.
Duality for finite Hilbert algebras.
Discrete Math., 305(1-3):74–99, 2005.

S. A. Celani and R. Jansana.
On the free implicative semilattice extension of a Hilbert algebra.
MLQ Math. Log. Q. **58** (2012), no. 3, 188–207.

N. G. de Bruijn.
Exact finite models for minimal propositional calculus over a finite alphabet.
Technical Report EUT Report 75-WSK-02, Eindhoven University of Technology, Eindhoven, The Netherlands, 1975.

P. Köhler.
Brouwerian semilattices.
Trans. Amer. Math. Soc., 268(1):103–126, 1981.

K. Słomczyńska.
Unification and projectivity in Fregean varieties.
Log. J. IGPL, 20(1):73–93, 2012.

K. Słomczyńska.
Free equivalential algebras.
Ann. Pure Appl. Logic, 155(2):86–96, 2008.

A. Urquhart.
Implicational formulas in intuitionistic logic.
J. Symbolic Logic, 39:661–664, 1974.