Esakia duality for temporal Heyting algebras

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Presentation overview

1. Preliminaries

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2. Esakia duality for **tHA**

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3. Notions of reachability

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4. Characterisations

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4. Characterisations
5. Applications to **tHC**

Preliminaries

The algebras

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- $\langle A, \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra,
- $\Box(a \wedge b) = \Box a \wedge \Box b$,
- $a \leqslant \Box a$,
- $\Box a \leqslant b \vee (b \to a)$.

The algebras (cont.)

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Fact — Esakia, 2006

The lattice of open sets of a topological space is a frontal Heyting algebra where $\Box$ is the *coderivative* operator.

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$$\begin{aligned}\Box U &:= \{x \in X \mid x \text{ is a } \textit{frontal point} \text{ of U}\} \\ &:= \{x \in X \mid \exists V \in \Omega \,.\, x \in V \text{ and } V \subseteq U \cup \{x\}\}\end{aligned}$$

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Fact — Kuznetsov, 1985

If one adds the algebraic version of the "intuitionistic Gödel-Lob formula"

$$\Box a \to a \leqslant a,$$

one arrives at the variety of *frontons* (also called *KM-algebras*), which generate the variety **HA**.

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A *temporal Heyting algebra* is an algebra $\langle A, \wedge, \vee, \rightarrow, \blacklozenge, \square, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee, \rightarrow, \square, 0, 1\rangle$ is a frontal Heyting algebra
- $\blacklozenge a \leqslant b$ if and only if $a \leqslant \square b$.

Typical (classical) case

	Past	Future
$\exists$	$\blacklozenge$	$\lozenge$
$\forall$	$\blacksquare$	$\square$

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Typical (classical) case

	Past	Future
$\exists$	$\blacklozenge$	$\Diamond$
$\forall$	$\blacksquare$	$\Box$

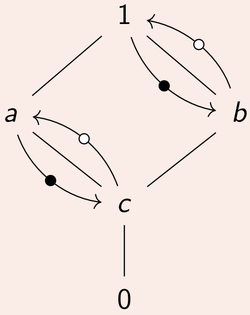
Note

No $\Diamond$ or $\blacksquare$ and no way to define them without classical negation.

The algebras (cont.)

Examples : temporal Heyting algebras

Where □ is empty and ⧫ is filled :



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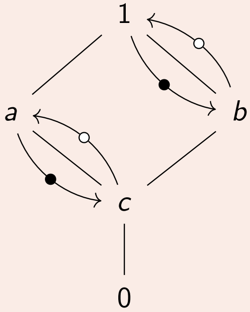

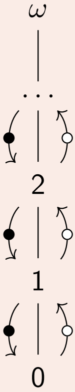

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Intuition : transits

R lies somewhere between $<$ and $\leqslant$, so it can be thought of as $\leqslant$ missing *some* reflexive loops.

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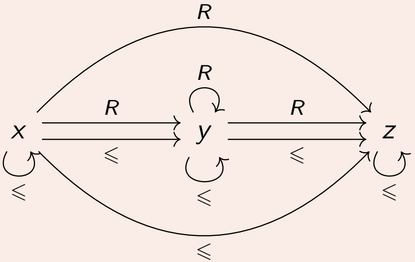

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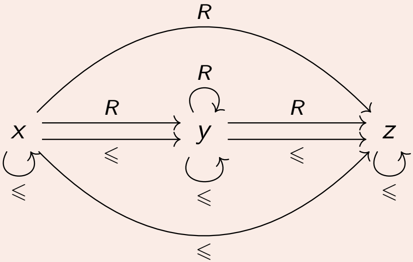

Note that x and z lack reflexive loops (for R).

The frames (cont.)

Example : transits

$$\begin{tikzcd}
x \arrow[loop below, "\leqslant"] \arrow[r, "R"] \arrow[r, shift right, "\leqslant"'] \arrow[rr, bend left, "R"] \arrow[rr, bend right, "\leqslant"'] & y \arrow[loop above, "R"] \arrow[loop below, "\leqslant"] \arrow[r, "R"] \arrow[r, shift right, "\leqslant"'] & z \arrow[loop below, "\leqslant"]
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$$
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x \arrow[r, shift left, "R"] \arrow[r, shift right, "\leqslant"'] \arrow[loop below, "\leqslant"] \arrow[rr, bend left=60, "R"] \arrow[rr, bend right=60, "\leqslant"'] & y \arrow[r, shift left, "R"] \arrow[r, shift right, "\leqslant"'] \arrow[loop above, "R"] \arrow[loop below, "\leqslant"] & z \arrow[loop below, "\leqslant"]
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Intuition : temporal transits

Temporal transits are simply transits with an explicit backwards relation.

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Intuition : temporal Esakia spaces

Temporal Esakia spaces are the natural topologisation of temporal transits, which are themselves the natural temporalisation of transits.

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Temporal Esakia morphism

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- $fx_2 \; R^\circ \; y$ implies $\exists x_1 \in \mathbb{X}$ such that $x_2 \; R^\circ \; x_1$ and $y \leqslant fx_1$.

$$\begin{tikzcd}
x_2 \arrow[rr] \arrow[d, dashed, "R^\circ"'] & & fx_2 \arrow[dd, "R^\circ"] \\
x_1 \arrow[r] & fx_1 & \\
& & y \arrow[ul, dashed, "\leqslant"]
\end{tikzcd}$$

Esakia duality for tHA

Equivalence

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Theorem — Esakia, 1974

The category of Heyting algebras is dually equivalent to the category of Esakia spaces.

$$\mathbf{HA} \cong \mathbf{ES}^{\mathrm{op}}$$

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$$\mathbf{fHA} \cong \mathbf{fES}^{\mathrm{op}}$$

Equivalence (cont.)

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Duality for **fHA**

Given $\mathbb{A} \in$ **fHA**, we define R on the dual space in the standard way :

$$x\,R\,y \quad :\Longleftrightarrow \quad \forall a \in \mathbb{A}\,.\,\Box a \in x \Rightarrow a \in y$$

Equivalence (cont.)

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Given $\mathbb{X} \in$ **fES**, we define $\Box$ on the dual algebra in the standard way :

$$\begin{aligned} \Box K &:= -R^{-1}[-K] \\ &:= \{x \in \mathbb{X} \mid R[x] \subseteq K\} \end{aligned}$$

Equivalence (cont.)

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Duality for tHA

Given $\mathbb{A} \in \mathbf{tHA}$, we define R° on the dual space in the standard way :

$$y \; R^\circ \; x \quad :\Longleftrightarrow \quad \forall a \in \mathbb{A} \, . \, a \in x \Rightarrow \blacklozenge a \in y$$

Equivalence (cont.)

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Given $\mathbb{X} \in$ **tES**, we define $\blacklozenge$ on the dual algebra in the standard way :

$$\begin{aligned} \blacklozenge K &:= R[K] \\ &:= \{x \in \mathbb{X} \mid R^\circ[x] \cap K \neq \varnothing\} \end{aligned}$$

Equivalence (cont.)

Duality for tHA

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Theorem — DQA, 2025

The category of temporal Heyting algebras is dually equivalent to the category of temporal Esakia spaces.

$$\mathbf{tHA} \cong \mathbf{tES}^{\mathrm{op}}$$

Further duality theory

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Theorem

Given $\mathbb{A} \in \mathbf{HA}$,

$$\langle \mathsf{Cong}(\mathbb{A}), \subseteq \rangle \cong \langle \mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong \langle \mathsf{ClUp}(\mathbb{A}_*), \supseteq \rangle$$

Further duality theory

Theorem

Given $\mathbb{A} \in$ **HA**,

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Intuition

Filters are surjective **HA**-morphisms *out* of $\mathbb{A}$ and closed upsets are injective **ES**-morphisms *into* $\mathbb{A}_*$.

Further duality theory (cont.)

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Definition : $\blacklozenge$-filter

A $\blacklozenge$-*filter* is a filter $F \subseteq \mathbb{A}$ such that

$$a \to b \in F \quad \Longrightarrow \quad \blacklozenge a \to \blacklozenge b \in F.$$

Further duality theory (cont.)

Definition : ⧫-filter

A ⧫-*filter* is a filter $F \subseteq \mathbb{A}$ such that

$$a \to b \in F \quad \Longrightarrow \quad \blacklozenge a \to \blacklozenge b \in F.$$

Intuition : ⧫-filter

A ⧫-filter is analogous to an *open filter* (also called a *"modal filter"*) on a modal algebra (also called a "Boolean algebra with operators").

Further duality theory (cont.)

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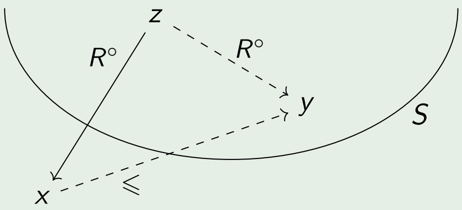
Definition : Archival subset

An *archival* subset is a subset $S \subseteq \mathbb{X}$ such that



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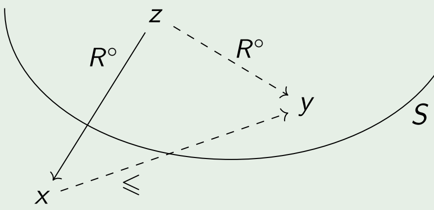

Intuition : Archival subset

An archival subset is analogous to a (generated) subframe.

Further duality theory (cont.)

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Theorem DQA, 2025

Given $\mathbb{A} \in \mathbf{tHA}$,

$$\langle \mathsf{Cong}(\mathbb{A}), \subseteq \rangle \cong \langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong \langle \mathsf{ClArcUp}(\mathbb{A}), \supseteq \rangle$$

Further duality theory (cont.)

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Definition : $\blacklozenge$-compatible element

A $\blacklozenge$-*compatible element* is an element $a \in \mathbb{A}$ such that

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Intuition : $\blacklozenge$-compatible element

$\blacklozenge$-compatible elements are analogous to *open elements* of an interior algebra.

Further duality theory (cont.)

Definition : ⧫-compatible element

A ⧫-*compatible element* is an element $a \in \mathbb{A}$ such that

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Intuition : ⧫-compatible element

⧫-compatible elements are analogous to *open elements* of an interior algebra.

Fact DQA, 2025

Given $\mathbb{A} \in \mathbf{tHA}_{\text{fin}}$,

$$\langle \mathrm{Cong}(\mathbb{A}), \subseteq \rangle \cong \langle \blacklozenge\mathrm{Filt}(\mathbb{A}), \subseteq \rangle \cong \langle \blacklozenge\mathrm{Com}(\mathbb{A}), \geqslant \rangle$$

Notions of reachability

Topo-reachability

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Definition : Topo-reachability

Given $\mathbb{X} \in \mathbf{tES}$,

$$x \trianglelefteq y \quad :\Longleftrightarrow \quad y \in \bigcap \{ C \in \mathrm{ClArcUp}(\mathbb{X}) \mid x \in C \}$$

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Intuition: Topo-reachability

Topo-reachability analogous to well-known "specialisation ordering" but ranges over **closed archival upsets** instead of just **closed** sets.

Z-reachability

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Intuition: Z-reachability

"Zig-zagging" down until you hit a minimal or R-reflexive point and back up via $\leqslant$ a finite number of times.

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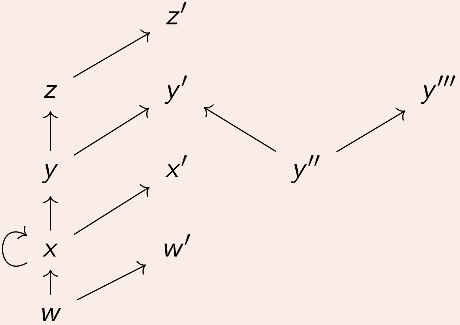
Example : Z-reachability



$$Z_0[z] = \{z\}$$
$$B[Z_0[z]] = \{z, x, y\}$$
$$Z_1[z] = \{z, x, y, x', y', z'\}$$
$$B[Z_1[z]] = \{z, x, y, x', y', z', y''\}$$
$$Z_2[z] = \{z, x, y, x', y', z', y'', y'''\}$$
$$\dots$$
$$Z[z] = \{z, x, y, x', y', z', y'', y'''\}$$

Coincidence

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Proposition — DQA, 2025

Given $\mathbb{X} \in \mathbf{tES}_{\mathrm{fin}}$,

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Coincidence

Proposition DQA, 2025

Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$x \trianglelefteq y \quad \Longleftrightarrow \quad x \, Z \, y .$$

Intuition

The two notions of reachability agree in the finite case.

Characterisations

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Following Venema, 2004, we give lattice-theoretic and dual order-topological characterisations of simple and subdirectly-irreducible members of our variety.

Simple algebras

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Theorem DQA, 2025

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Simple algebras

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Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent :

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- $\mathbb{A}_+$ is Z-rooted.

Applications to **tHC**

The temporal Heyting calculus

The temporal Heyting calculus

Definition

$$\varphi := p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \blacklozenge\varphi \mid \bot \mid \top$$

The temporal Heyting calculus

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Definition

$$(\mathrm{MP})\ \frac{\varphi \quad \varphi \to \chi}{\chi} \qquad (\mathrm{US})\ \frac{\varphi}{\varphi[\chi/p]} \qquad (\mathrm{PD})\ \frac{\varphi \to \chi}{\blacklozenge\varphi \to \blacklozenge\chi}$$

The temporal Heyting calculus (cont.)

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Definition : mHC Esakia, 2006

$$
\begin{aligned}
\mathbf{mHC} := \mathbf{IPC} & \\
& \oplus \Box(p \to q) \to (\Box p \to \Box q) \\
& \oplus p \to \Box p \\
& \oplus \Box p \to (q \vee q \to p) \quad \text{with (US) and (MP)}
\end{aligned}
$$

The temporal Heyting calculus (cont.)

Definition : **mHC** — Esakia, 2006

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Definition : **tHC** — Esakia, 2006

$$\begin{aligned} \mathbf{tHC} := \mathbf{mHC} & \\ & \oplus \blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q) \\ & \oplus \blacklozenge\bot \to \bot \\ & \oplus p \to \Box\blacklozenge p \\ & \oplus \blacklozenge\Box p \to p \quad \text{with (US) and (MP) and (PD)} \end{aligned}$$

Completeness

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Theorem DQA, 2025

The logic **tHC** is complete with respect to **tHA**.

Completeness

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The logic **tHC** is complete with respect to **tES**.

Stronger completeness results

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Theorem

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Stronger completeness results

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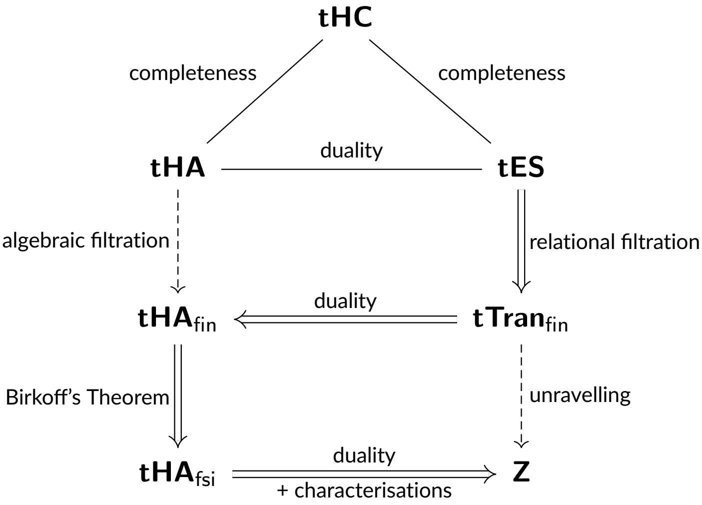
The logic **IPC** is complete with respect to the class of finite, rooted posets.

Theorem DQA, 2025

The logic **tHC** is complete with respect to the class of finite, Z-rooted temporal transits (denoted by **Z**).

Stronger completeness results (cont.)



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Thank you !

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DQ Alvarez, 2025