Regular epis in the regular category of rational polyhedra and $\mathbb{Z}$ maps

Luca Spada

Dipartimento di Matematica — Università di Salerno

joint work with M. Menni

TACL 2026, Krakow 30 July 2026

Key question

Can a category of spaces have a good theory of congruences?

Key question

Can a category of spaces have a good theory of congruences?
$=$ First Isomorphism Theorem $+$ good composition of congruences.

Key question

Can a category of spaces have a good theory of congruences?
$=$ First Isomorphism Theorem $+$ good composition of congruences.

Categories of spaces rarely enjoy this property.

Key question

Can a category of spaces have a good theory of congruences?
$=$ First Isomorphism Theorem $+$ good composition of congruences.

Categories of spaces rarely enjoy this property.

We will see that rational polyhedra do have this feature.

The story

Today we will see that:

1. Ordinary polyhedra inherit this feature form **Set**;
2. The case of rational polyhedra is different;
3. Replacing **Set** by a richer category restores this inheritance.

Outline

Regularity

Regular epis

An arrow is called regular epi if it is the coequalizer of some pair of maps:

$$
\begin{tikzcd}
R \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, two heads, "q"] & Y
\end{tikzcd}
$$

Intuition

Regular epis should be thought of as quotients over the equivalence relation induced by f_1 and f_2.

Regular categories

A regular category is a category in which the first isomorphism theorem holds (functorially).

Regular categories

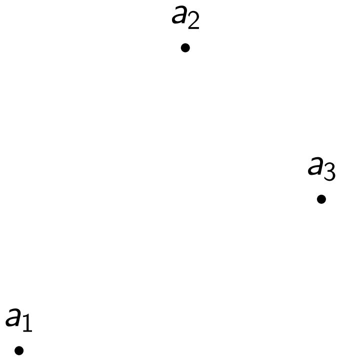
A regular category is a category in which the first isomorphism theorem holds (functorially).

Regular categories

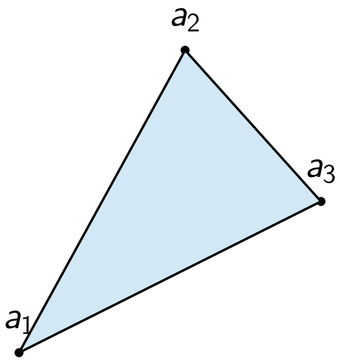
A category $\mathcal{C}$ is regular if:

- it has finite limits;
- every arrow factors as regular epi $+$ mono (image factorization);
- regular epis are stable under pullback.

Convex combinations

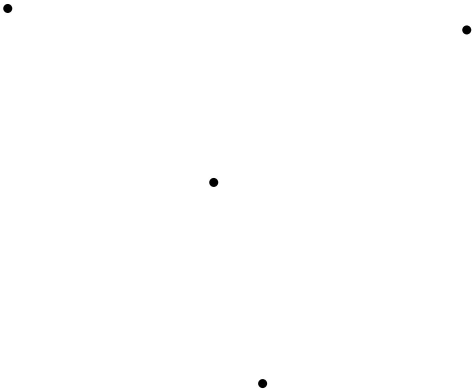


Convex combinations

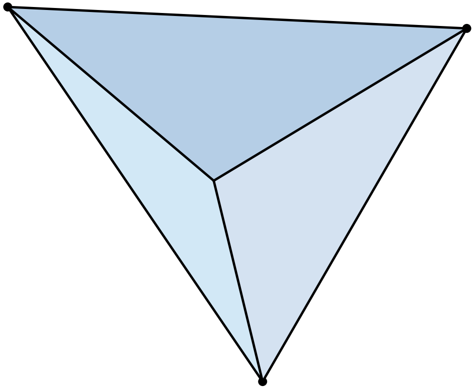


$$\left\{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \mid \lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 1\right\}$$

Convex combinations

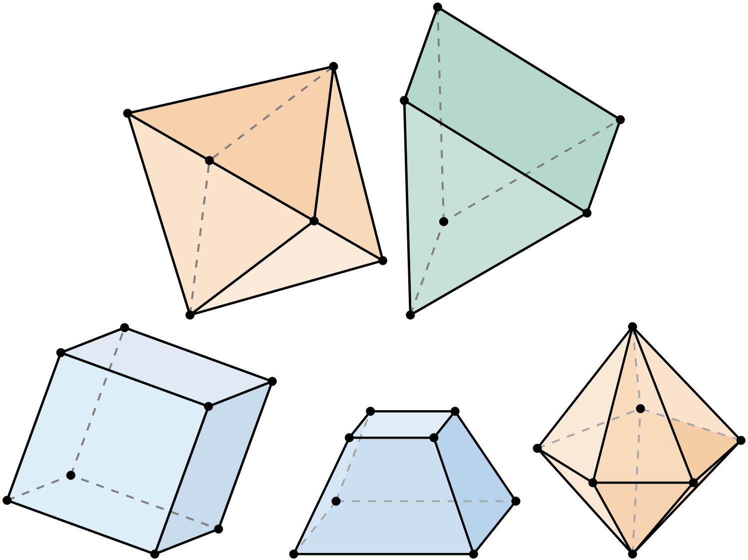


Convex combinations



$$\left\{ \sum_{i=1}^{n} \lambda_i a_i \mid \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

Polytopes

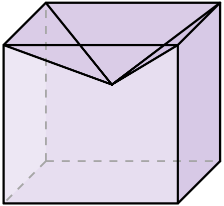


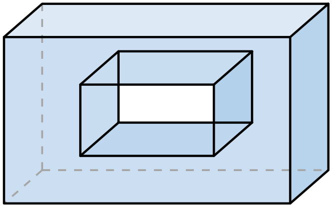
Affine maps

Affine maps (i.e. maps preserving affine combinations) take polytopes into polytopes.

Thus, we have a category $\mathcal{L}$ of polytopes and (restrictions of) affine maps between them.

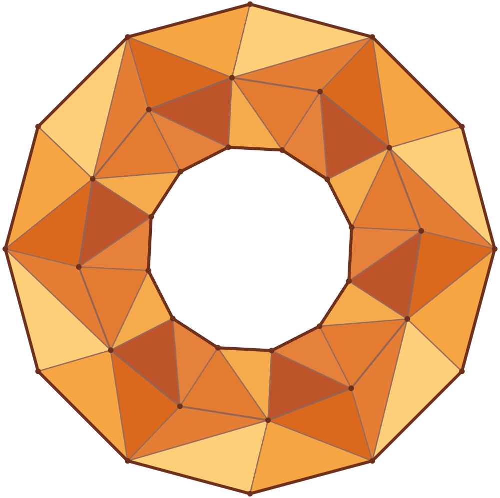
Polyhedra



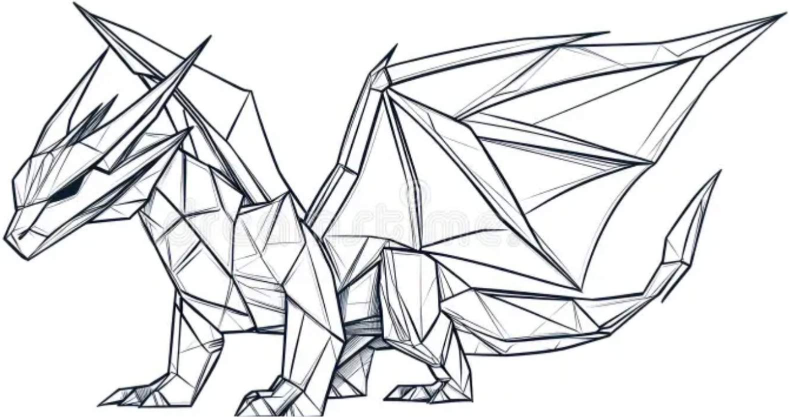


Polyhedron = finite union of polytopes.

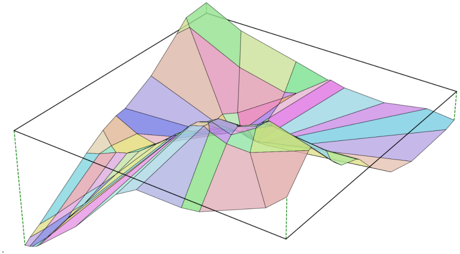
Qbwarzanek bread



Wawel Dragon



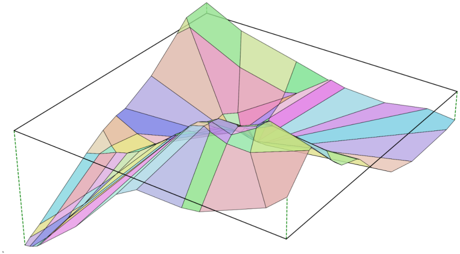
Piecewise linear maps



A continuous map $f : A \to B$ is piecewise (affinely) linear if there are finitely many affine maps $f_1, \ldots, f_\ell$ such that

$$\text{for every } x \in A \text{ there is } 1 \leq i \leq \ell \quad f(x) = f_i(x).$$

Piecewise linear maps

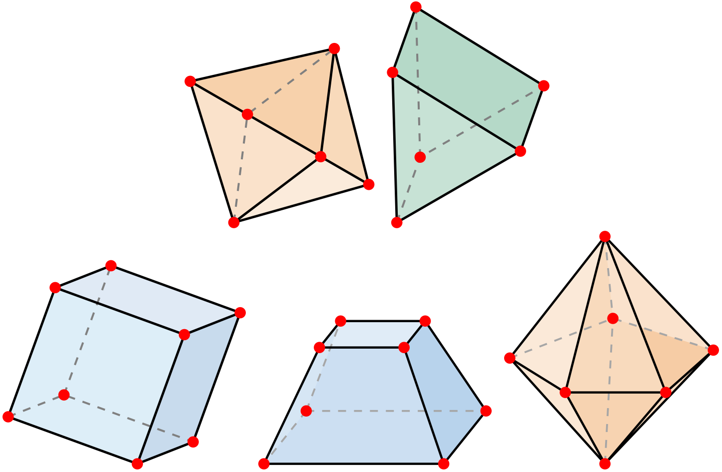


A continuous map $f: A \to B$ is piecewise (affinely) linear if there are finitely many affine maps $f_1, \ldots, f_\ell$ such that

$$\text{for every } x \in A \text{ there is } 1 \leq i \leq \ell \quad f(x) = f_i(x).$$

$\mathcal{PL}$: the category of polyhedra and PL maps between them.

Rational polytopes



The vertices • have coordinates in $\mathbb{Q}$.

The category $\mathcal{PL}_{\mathbb{Z}}$

A rational polyhedron is a finite union of rational polytopes.

The category $\mathcal{PL}_{\mathbb{Z}}$

A rational polyhedron is a finite union of rational polytopes.

A $\mathbb{Z}$ map is a PL map whose affine pieces have integer coefficients.

The category $\mathcal{PL}_{\mathbb{Z}}$

A rational polyhedron is a finite union of rational polytopes.

A $\mathbb{Z}$ map is a PL map whose affine pieces have integer coefficients.

$\mathcal{PL}_{\mathbb{Z}}$: category of rational polyhedra and $\mathbb{Z}$ maps between them.

Three categories

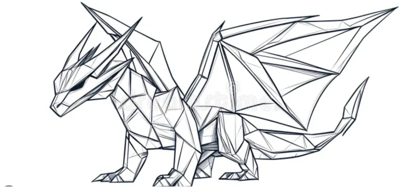
So, we have:

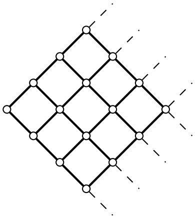
Category	Objects	Arrows
$\mathcal{L}$	polytopes	affine maps
$\mathcal{PL}$	polyhedra	piecewise affinely linear maps
$\mathcal{PL}_{\mathbb{Z}}$	rational polyhedra	$\mathbb{Z}$ maps

Stone duality for MV-algebras

FinSet	dual to	Finitely presented Boolean algebras
$\mathcal{PL}$	dual to	Finitely presented Riesz MV-algebras
$\mathcal{PL}_{\mathbb{Z}}$	dual to	Finitely presented MV-algebras
$\mathcal{L}$	dual to	??

Stone duality for MV-algebras





Wawel Dragon is dual to some f.p. MV-algebra.

Outline

The title

The problem

The result

The key idea

The abstract setting

The problem

Our aim is now to:

- Prove that $\mathcal{PL}_{\mathbb{Z}}$ is regular,
  - finite limits,
  - image factorisation,
  - pull-back stable regular epis.
- Characterise regular epis in $\mathcal{PL}_{\mathbb{Z}}$.

The case of $\mathcal{PL}$

To prove that $\mathcal{PL}$ is regular and to characterise its regular epis, the underlying-set functor

$$\mathcal{PL} \longrightarrow \mathbf{Set}$$

does most of the work.

- Limits are calculated in **Set**,
- Image factorisations are calculated in **Set**,
- **Set** is regular and its regular epis are just surjective functions.

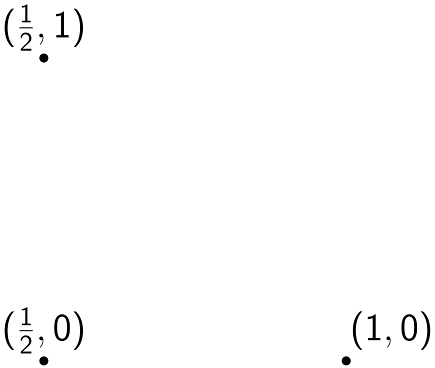
$\therefore$ $\mathcal{PL}$ is regular and its regular epis are just surjective maps.

Forgetting in Set does not work in $\mathcal{PL}_{\mathbb{Z}}$

$(\frac{1}{2}, 1)$

$(\frac{1}{2}, 0)$

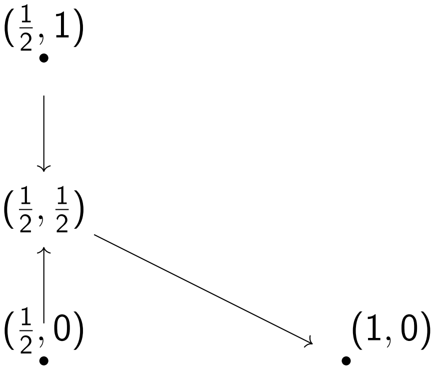
Forgetting in Set does not work in $\mathcal{PL}_{\mathbb{Z}}$



Forgetting in Set does not work in $\mathcal{PL}_{\mathbb{Z}}$

$$
\begin{tikzcd}
(\frac{1}{2}, 1) \arrow[dr, swap, "f = (2x, 0)"] & \\
(\frac{1}{2}, 0) \arrow[r] & (1, 0)
\end{tikzcd}
$$

Forgetting in Set does not work in $\mathcal{PL}_{\mathbb{Z}}$



Forgetting in Set does not work in $\mathcal{PL}_{\mathbb{Z}}$

$$
\begin{tikzcd}
(\frac{1}{2}, 1) \arrow[d] & \\
(\frac{1}{2}, \frac{1}{2}) \arrow[dr] & \\
(\frac{1}{2}, 0) \arrow[u] & (1, 0) \arrow[ul, bend left, "\times"]
\end{tikzcd}
$$

The obstruction is not geometric: the integer coefficients introduce arithmetic obstructions.

An invariant for points: denominators

To any point $x \in \mathbb{R}^n$ we assign a denominator in $\mathbb{N}$.

$$\operatorname{den} x := \begin{cases} \operatorname{lcm}\{\operatorname{den} x_i \mid i \leq n\} & \text{if } x \in \mathbb{Q}^n, \\ 0 & \text{if } x \notin \mathbb{Q}^n. \end{cases}$$

An invariant for points: denominators

To any point $x \in \mathbb{R}^n$ we assign a denominator in $\mathbb{N}$.

$$\operatorname{den} x := \begin{cases} \operatorname{lcm}\{\operatorname{den} x_i \mid i \leq n\} & \text{if } x \in \mathbb{Q}^n, \\ 0 & \text{if } x \notin \mathbb{Q}^n. \end{cases}$$

Key observation

$\mathbb{Z}$ maps decrease denominators in the sense that

$$\boxed{\operatorname{den} f(x) \text{ divides } \operatorname{den} x.}$$

Back to the example

$$
\begin{tikzcd}
\mathrm{den}(\tfrac{1}{2}, 1) = 2 \arrow[d] & \\
\mathrm{den}(\tfrac{1}{2}, \tfrac{1}{2}) = 2 \arrow[dr] & \\
\mathrm{den}(\tfrac{1}{2}, 0) = 2 \arrow[u] & \mathrm{den}(1, 0) = {\color{teal}1} \arrow[ul, bend left, "\times" description]
\end{tikzcd}
$$

Outline

The result

Theorem

The category $\mathsf{PL}_{\mathbb{Z}}$ *is regular—in fact coherent and extensive.*

The result

Theorem

The category $\mathrm{PL}_{\mathbb{Z}}$ *is regular—in fact coherent and extensive.*

Theorem

A map f *is regular epi in* $\mathrm{PL}_{\mathbb{Z}}$ *iff it is surjective and*

$$\mathrm{den}\,(y) = \mathrm{gcd}\{\mathrm{den}(x) \mid f(x) = y\}.$$

Outline

Replace Set with a richer category

We need to enrich **Set** with enough arithmetical information to account for the integer coefficients in $\mathbb{Z}$ maps.

Idea

Decorate every point with an arithmetic invariant: a "denominator map" $d\colon X \to \mathbb{N}$ encoding some arithmetic of rational points.

The new ambient space $\mathbf{Set}_{\Omega}$

Let $(\Omega, \leq)$ be a partially ordered set.

The new ambient space $\mathbf{Set}_\Omega$

Let $(\Omega, \leq)$ be a partially ordered set.

Definition (The category $\mathbf{Set}_\Omega$)

Objects are pairs (X, d) with $d : X \to \Omega$.

The new ambient space $\mathbf{Set}_\Omega$

Let $(\Omega, \leq)$ be a partially ordered set.

Definition (The category $\mathbf{Set}_\Omega$)

Objects are pairs (X, d) with $d : X \to \Omega$.
Morphisms $f\colon (X, d_X) \to (Y, d_Y)$ are functions $f\colon X \to Y$ such that

$$d_X(x) \leq d_Y(f(x)) \quad \forall x \in X.$$

The new ambient space: $\mathbf{Set}_{\mathbb{N}}$

Lemma (Folklore)

If Ω is a frame (=complete lattice, finite meets distribute over arbitrary joins), then $\mathbf{Set}_{\Omega}$ is a quasitopos (thus, a regular category).

The new ambient space: $\mathbf{Set}_{\mathbb{N}}$

Lemma (Folklore)

If Ω is a frame (=complete lattice, finite meets distribute over arbitrary joins), then $\mathbf{Set}_\Omega$ is a quasitopos (thus, a regular category).

A morphism $f : (X, d_X) \to (Y, d_Y)$ in $\mathbf{Set}_\Omega$ is a regular epi iff

- *the underlying function $f : X \to Y$ is surjective, and*
- *for all $y \in Y$,*

$$d_Y(y) = \bigvee \{d_X(x) \mid f(x) = y\}.$$

The divisibility lattice

Lemma (Folklore)

The divisibility order induces on $\mathbb{N}$ the structure of a coframe, i.e., the lattice $(\mathbb{N}, \mathrm{gcd}, \mathrm{lcm})^{op}$ is a frame.

The divisibility lattice

Lemma (Folklore)

The divisibility order induces on $\mathbb{N}$ the structure of a coframe, i.e., the lattice $(\mathbb{N}, \mathrm{gcd}, \mathrm{lcm})^{op}$ is a frame.

Thus, $\mathbf{Set}_{\Omega_{\mathbb{N}}}$ with $\Omega_{\mathbb{N}} = (\mathbb{N}, \text{reverse div})$ is a quasitopos. In particular, it is a regular category and the regular epis are exactly the surjective functions such that for all $y \in Y$,

$$d_Y(y) = \mathrm{gcd}\{d_X(x) \mid f(x) = y\}.$$

Outline

From the example to the general method

$$\begin{tikzcd} \mathcal{PL} \arrow[d] \\ \text{Forget into } \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{PL}_{\mathbb{Z}} \arrow[d] \\ \text{Forget into } \mathbf{Set}_{\Omega_{\mathbb{N}}} \end{tikzcd}$$

Is there a common mechanism?

Can regularity be lifted from a suitable ambient category to a category of geometrically definable spaces?

Setup: a faithful functor $\mathcal{T} \to \mathcal{S}$

Fix:

- $\mathcal{T}$ a category with finite products
  (e.g. affine spaces with affine maps);

Setup: a faithful functor $\mathcal{T} \to \mathcal{S}$

Fix:

- $\mathcal{T}$ a category with finite products
  (e.g. affine spaces with affine maps);
- $\mathcal{S}$ a regular category;
  (e.g. **Set**)

Setup: a faithful functor $\mathcal{T} \to \mathcal{S}$

Fix:

- $\mathcal{T}$ a category with finite products
  (e.g. affine spaces with affine maps);
- $\mathcal{S}$ a regular category;
  (e.g. **Set**)
- $\mathcal{T} \to \mathcal{S}$ a faithful, finite-product preserving functor.
  (e.g. forgetful)

Definition: system of blocks

System of blocks B over $\mathcal{T} \to \mathcal{S}$

A family of monomorphisms $S \hookrightarrow A$ in $\mathcal{S}$ (A ambient) closed under:

Definition: system of blocks

System of blocks B over $\mathcal{T} \to \mathcal{S}$

A family of monomorphisms $S \hookrightarrow A$ in $\mathcal{S}$ (A ambient) closed under:

1. isomorphisms (Blocks are subobjects);

Definition: system of blocks

System of blocks B over $\mathcal{T} \to \mathcal{S}$

A family of monomorphisms $S \hookrightarrow A$ in $\mathcal{S}$ (A ambient) closed under:

1. isomorphisms (Blocks are subobjects);
2. pullbacks (Blocks are closed under intersection);

Definition: system of blocks

System of blocks B over $\mathcal{T} \to \mathcal{S}$

A family of monomorphisms $S \hookrightarrow A$ in $\mathcal{S}$ (A ambient) closed under:

1. isomorphisms (Blocks are subobjects);
2. pullbacks (Blocks are closed under intersection);
3. images along affine maps.

Definition: system of blocks

System of blocks B over $\mathcal{T} \to \mathcal{S}$

A family of monomorphisms $S \hookrightarrow A$ in $\mathcal{S}$ (A ambient) closed under:

1. isomorphisms (Blocks are subobjects);
2. pullbacks (Blocks are closed under intersection);
3. images along affine maps.

- Think: “basic definable sets” inside ambient spaces.

The category $\mathrm{Sp}(\mathrm{B})$ of definable spaces and maps

Objects of $\mathrm{Sp}(\mathrm{B})$

Are B-blocks: $\mu_X \colon \mathrm{P}X \hookrightarrow \mathrm{A}X$ (with $\mathrm{A}X$ ambient).

The category $\mathrm{Sp}(\mathrm{B})$ of definable spaces and maps

Objects of $\mathrm{Sp}(\mathrm{B})$

Are B-blocks: $\mu_X \colon \mathrm{P}X \hookrightarrow \mathrm{A}X$ (with $\mathrm{A}X$ ambient).

Morphisms $f \colon X \to Y$

Maps $f \colon \mathrm{P}X \to \mathrm{P}Y$ in $\mathcal{S}$ such that the graph

$$\langle \mu_X, \mu_Y f \rangle \colon \mathrm{P}X \longrightarrow \mathrm{A}X \times \mathrm{A}Y$$

is a block.

The category $\mathrm{Sp}(\mathrm{B})$ of definable spaces and maps

Objects of $\mathrm{Sp}(\mathrm{B})$

Are B-blocks: $\mu_X \colon \mathrm{P}X \hookrightarrow \mathrm{A}X$ (with $\mathrm{A}X$ ambient).

Morphisms $f \colon X \to Y$

Maps $f \colon \mathrm{P}X \to \mathrm{P}Y$ in $\mathcal{S}$ such that the graph

$$\langle \mu_X, \mu_Y f \rangle \colon \mathrm{P}X \longrightarrow \mathrm{A}X \times \mathrm{A}Y$$

is a block.

Remark: The graph condition is intrinsic. In the applications, we must prove that it recovers the familiar geometric maps.

Product condition

A remarkable point is that one elementary closure condition is enough.

Product condition

A remarkable point is that one elementary closure condition is enough.

Product condition on B

A system of blocks B over $\mathcal{T} \to \mathcal{S}$ satisfies the product condition if

1. id_1 is a block (1=terminal object) ;
2. products of blocks are blocks.

Main lifting result: $\mathrm{Sp}(\mathbb{B})$ is regular

Theorem

If $\mathbb{B}$ satisfies the product condition, then:

- $\mathsf{Sp}(\mathbb{B})$ *is a regular category;*
- *the functor* $\mathsf{Sp}(\mathbb{B}) \to \mathcal{S}$ *builds finite limits and preserves and reflects regular epis.*

Proof idea

Finite limits come from products and graph intersections; image factorizations and regular epis are inherited from $\mathcal{S}$.

Polyblocks

Definition

A B-polyblock is a finite join of B-blocks.

Let B' be the system of blocks whose blocks are polyblocks.

Remark

If B satisfies product condition, so does B'.

Application 1: polytopes are regular

Set

- $\mathcal{S} = \mathbf{Set}$,
- $\mathcal{T}$ = finite-dimensional real affine spaces with affine maps,
- B = polytopes as subobjects.

Lemma

A function between two polytopes can be extended to an affine map between the two ambient spaces iff its graph is a polytope.
Thus, $\mathrm{Sp}(\mathrm{B})$ *is the category of polytopes and affine linear maps between them.*

Theorem

The system B *satisfies the product condition. Hence,* $\mathrm{Sp}(\mathrm{B})$ *is regular.*
The regular epis are simply the surjective maps.

Application 2: polyhedra are regular

Consider now B', they are polyhedra (=finite unions of polytopes).

Lemma

A function between two polyhedra can be extended to a PL map between the two ambient spaces iff its graph is a polyhedron. Thus, $\mathrm{Sp}(\mathrm{B}')$ *is the category of polyhedra and PL maps between them.*

Theorem

The category $\mathrm{Sp}(\mathrm{B}')$ *is regular. The regular epis are simply the surjective maps.*

Rational polytopes as blocks

Set

- $\mathcal{T}$ = the category of spaces with denominators $(\mathbb{R}^n, \mathrm{den})$,
- $\mathcal{S} = \mathbf{Set}_{\Omega_{\mathbb{N}}}$,
- B = rational polytopes.

Lemma (Joint work with V. Marra)

A function between two rational polytopes can be extended to an $\mathbb{Z}$-affine map between the two ambient spaces iff its graph is a polytope and it decreases denominators.

Then $\mathrm{Sp}(\mathrm{B})$ is the category of rational polytopes with $\mathbb{Z}$-affine maps.

Rational polyhedra and $\mathbb{Z}$ maps as polyblocks

Corollary

A PL map between rational polyhedra that decreases denominators is a $\mathbb{Z}$-map.

Consider $\mathrm{B}' =$ finite unions of rational polytopes. $\mathsf{Sp}(\mathrm{B}')$ is the category of rational polyhedra and $\mathbb{Z}$ maps

Theorem

The category $\mathsf{Sp}(\mathrm{B}')$ is regular. The regular epis are the surjective $\mathbb{Z}$ maps satisfying the condition:

$$\mathrm{den}\,(y) = \mathrm{gcd}\{\mathrm{den}(x) \mid f(x) = y\}.$$