Semiconical residuated lattices with an idempotent skeleton

Valeria Giustarini
joint work with Nick Galatos, Francesco Manfucci and Sara Ugolini

Artificial Intelligence Research Institute (IIIA, CSIC), Spain

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Introduction

Residuated lattices are algebraic structures originally arising from the study of ideals of rings.

These structures are very important in the field of algebraic logic since they constitute the equivalent algebraic semantics, in the sense of Blok and Pigozzi, of **substructural logics**.

These are a broad class of logics containing many of the interesting examples of logics amenable to the algebraic study: classical logic, intuitionistic logic, intermediate logics, many-valued logics, relevance logics to name a few.

Introduction

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- **integral** if 1 is the top of the lattice;
- **conical** if 1 is comparable with any other element of the lattice;
- **semiconical** if its is a subdirect product of conical ones.

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Also, categorical equivalences can help in the study and understanding the inner structure of the algebras in the class of structures we are considering.

Semiconical Skeleton-Idempotent residuated lattices

Galatos, Raftery and Fussner studied the variety of semiconical resiaduated lattices that are idempotent. Some of the main results that they obtained are:

- a characterization of the class of generators;
- a decomposition system for the generating class;
- a categorical equivalence for a subclass of semiconical idempotent resiaduated lattices, namely the Generalized Sugihara monoids.

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In this contribution, we would like to generalize their results dropping the hypothesis of full idempotency. In particular, we will consider semiconical residuated lattices having an idempotent skeleton.

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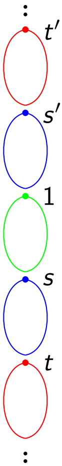
Let G_B be the class of algebras that pictorially look like this:



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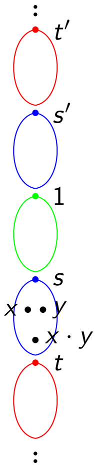

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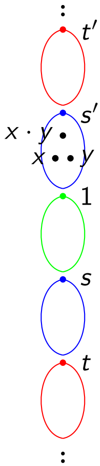

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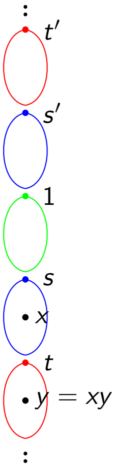

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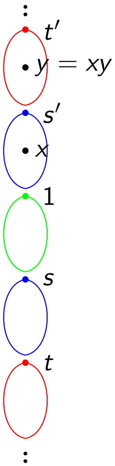

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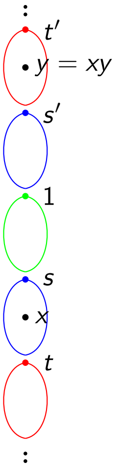

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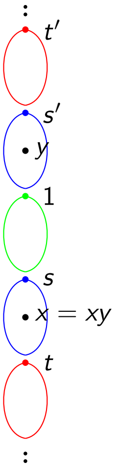

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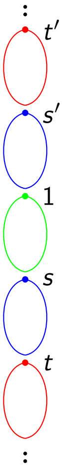

Let $B = V(G_B)$ which is the variety of **Semiconical Skeleton-Idempotent CRL**.

Semiconical Skeleton-Idempotent residuated lattices

So to be more precise, an algebra $\mathbf{A} \in \mathrm{G_B}$ if and only if it is such that:

- $x \to 1$ is a conical and idempotent element;
- if $x \leq 1 \leq y$, $xy = y$ or $xy = x$;
- if $x, y \leq 1$ and $(x \to 1) \to 1 \neq (y \to 1) \to 1$, $xy = x \wedge y$;
- if $x, y > 1$ and $(x \to 1) \to 1 \neq (y \to 1) \to 1$, $xy = x \vee y$;
- if $x, y > 1$ and $(x \to 1) \to 1 = (y \to 1) \to 1$ with $x \leq y$, then $((x \to y) \to 1) \to 1 = (x \to 1) \to 1$

Semiconical Skeleton-Idempotent residuated lattices

Thus, given $\mathbf{A} \in \mathsf{G_B}$, then

$$S = \{x \in A \mid x \to 1 = x\}$$

is a set of conical and idempotent elements and it is the domain of a Sugihara chain.

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Interestingly, this allows one to prove that if $a \in S$ is a negative γ-fixed point, then a is product irreducible. This fact is crucial since it gives us that the negative "bubbles" are indeed closed under lattice operations and products.

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Let $\mathbf{A} \in \mathsf{G_B}$.Then

1. S is the universe of a Sugihara chain $\mathbf{S}$ which is a subalgebra of $\mathbf{A}$;
2. every block $\gamma^{-1}(s)$ with $s \in S$ are prelattices having s as top. Moreover,
   1. if s has no lower cover, then $\gamma^{-1}(s)$ is a lattice;
   2. if $s = 1$, $\gamma^{-1}(1)$ is a CIRL where $x \to_1 y = (x \to y) \wedge 1$;
   3. if $s < 1$, $\gamma^{-1}(s)$ is a distributive lattice, a semigroup and the restriction of the implication to it, $\to_s$, is a partial operation, i.e. it is defined only between elements x, y with $x \not\leq y$;
   4. if $s > 1$, $\gamma^{-1}(s)$ is a semigroup, a partial distributive lattice, i.e. the meet operation may be a partial operation and join distributes over existing meets, and the restriction of the implication to it, $\to_s$, is a partial operations defined only between elements x and y with $x \leq y$.

Semiconical Skeleton-Idempotent residuated lattices

For the converse, let $\mathbf{S}$ be a Sugihara chain and let $\{\mathbf{A}_s \mid s \in \mathbf{S}\}$ a family of prelattices with top s such that:

1. if $s = 1$, $\mathbf{A}_1$ is a CIRL;
2. if $s < 1$, then $\mathbf{A}_s$ is a distributive lattice, a commutative semigroup and there is a partial operation $\rightharpoondown_s$, which is defined only between $x, y \in A_s$ with $x \not\leq y$, such that $x \rightharpoondown_s y = max\{z \in A_s \mid xz \leq y\}$;
3. if $s > 1$, then $\mathbf{A}_s$ is a partial distributive lattice, i.e. the join distributes over the existing meets, a commutative semigroup and there is a partial operation $\rightharpoonup_s$, defined only for $x, y \in A_s$ with $x \leq y$, such that $x \rightharpoonup_s y = max\{z \in A_s \mid xz \leq y\}$;
4. if $\mathbf{A}_s$ is not a lattice, then s has a lower cover $s^{\prec}$ in $\mathbf{S}$.

Let us call **decomposition system** the pair $\mathcal{D} = (\mathbf{S}, \{A_s \mid s \in S\})$.

Semiconical Skeleton-Idempotent residuated lattices

Given a decomposition system $\mathcal{D} = (\mathbf{S}, \{\mathbf{A}_s \mid s \in \mathbf{S}\})$, one can define the algebra $\mathbf{D} = \bigoplus_{s\in\mathbf{S}} \mathbf{A}_s$ having as domain $D = \bigoplus_{s\in S} A_s$ and for any $x \in A_s$ and $y \in A_t$

$$x \cdot y = \begin{cases} x \cdot_s y & \text{if } s = t \\ x & \text{if } s \neq t \text{ and } st = s \\ y & \text{if } s \neq t \text{ and } st = t \end{cases}$$

$$x \to y = \begin{cases} x \rightharpoondown_s y & \text{if } s = t \leq 1 \text{ and } x \not\leq y \\ x \rightharpoonup_s y & \text{if } s = t > 1 \text{ and } x \leq y \\ s' \vee y & \text{if } s < t \text{ or } s = t \leq 1 \text{ and } x \leq y \\ s' \wedge y & \text{if } t < s \text{ and } s = t \geq 1 \text{ and } x \not\leq y \end{cases}$$

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where $s' = s \to 1$.

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where $s' = s \to 1$. **It can be shown that the algebra $\mathbf{D} \in \mathsf{G_B}$.**

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Nicely, the generators of B can be described by means of positive universal sentence, relatively to CRL.

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Nicely, the generators of B can be described by means of positive universal sentence, relatively to CRL.

1. $\forall x, y(x' \leq y \text{ OR } y \leq x')$;
2. $\forall x((x')^2 = x')$;
3. $\forall x, y((y \vee 1) \leq (x \wedge 1)' \text{ OR } (x \wedge 1)(y \vee 1) = (y \vee 1))$;
4. $\forall x, y((x \wedge 1)' \leq (y \vee 1) \text{ OR } (x \wedge 1)(y \vee 1) = (x \wedge 1))$;
5. $\forall x((x \wedge 1)(x \wedge 1)' = (x \wedge 1))$;
6. $\forall x((x \vee 1)(x \vee 1)' = (x \vee 1)')$;
7. $\forall x, y((x \wedge 1)'' \leq (y \wedge 1)'' \text{ OR } (x \wedge 1)(y \wedge 1) = (y \wedge 1))$;
8. $\forall x, y((x \vee 1)'' \leq (y \vee 1)'' \text{ OR } (x \vee 1)(y \vee 1) = (x \vee 1))$;
9. $\forall x(x''' = x')$;
10. $\forall x, y((((x \vee 1) \to (((x \vee 1) \vee (y \vee 1)) \wedge (x \vee 1)''))'') = (x \vee 1)'')$

where $x' = x \to 1$.

Semiconical Skeleton-Idempotent residuated lattices

Thus the axiomatization of B, relatively to CRL, is obtained from the translations of the positive universal sentences describing its generating class.

1. $(x' \to y) \vee (y \to x') = 1$;
2. $(x')^2 = x'$;
3. $((y \vee 1) \to (x \wedge 1)') \vee ((y \vee 1) \leftrightarrow ((x \wedge 1)(y \vee 1))) = 1$;
4. $((x \wedge 1)' \to (y \vee 1)) \vee ((x \wedge 1) \leftrightarrow ((x \wedge 1)(y \vee 1))) = 1$;
5. $(x \wedge 1)(x \wedge 1)' = (x \wedge 1)$;
6. $(x \vee 1)(x \vee 1)' = (x \vee 1)'$;
7. $((x \wedge 1)'' \to (y \wedge 1)'') \to ((y \wedge 1) \to (x \wedge 1)(y \wedge 1)) = 1$;
8. $((x \vee 1)'' \to (y \vee 1)'') \vee ((x \vee 1) \to (x \vee 1)(y \vee 1)) = 1$;
9. $x''' = x'$;
10. $((((x \vee 1) \to (((x \vee 1) \vee (y \vee 1)) \wedge (x \vee 1)''))'')) \leftrightarrow (x \vee 1)'' = 1$

where $x' \leftrightarrow y = (x \to y) \wedge (y \to x)$.

A categorical equivalence for B

It is worth noticing that every element of an algebra $\mathbf{A} \in \mathsf{B}$, can be written as the product of its positive and negative component, i.e. $x = (x \wedge 1)(x \vee 1)$.

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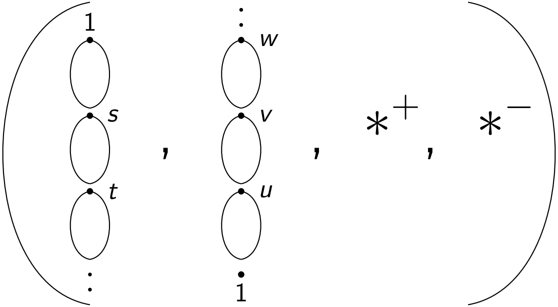
This fact suggest that there is a way to reconstruct the entire algebra **A** starting from its positive and negative cone.

Notice that since there can be different algebras **A** and **B** in B having the same positive and negative cone, it is also fundamental to consider the interaction between them. Thus what we need is to consider a class of **two sorted algebras**.

In particular, we will consider the variety of two sorted algebras generated by algebras that pictorially, looks like that:



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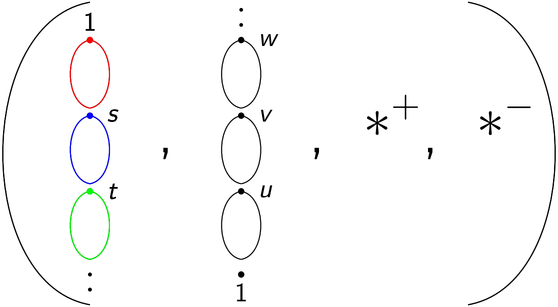

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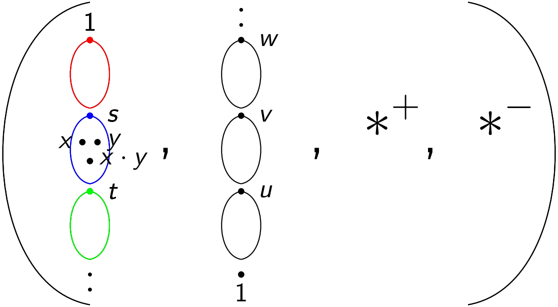

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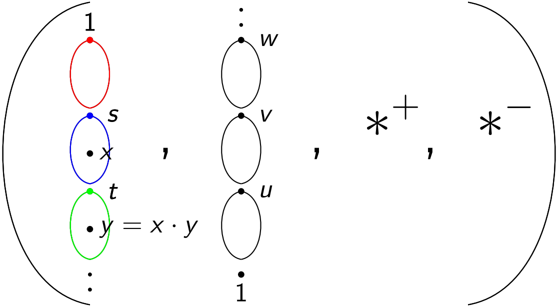

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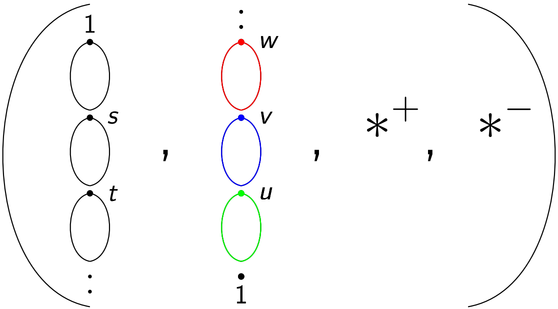

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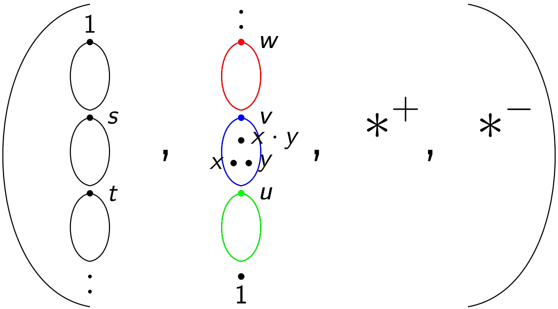

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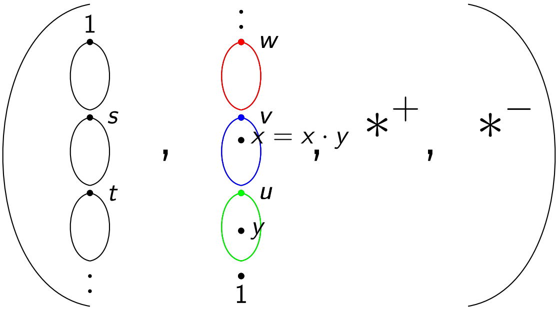

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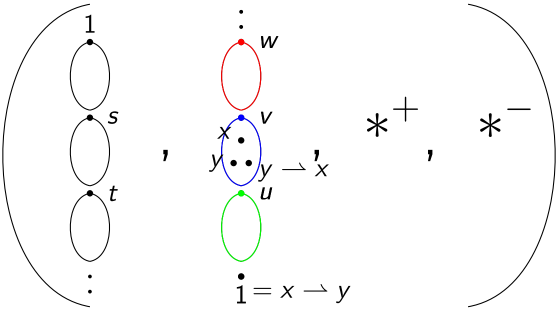

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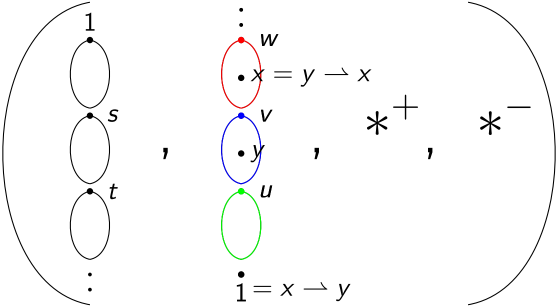

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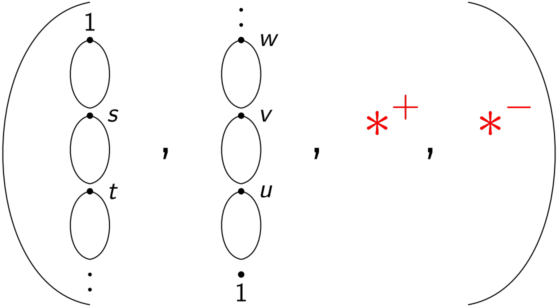

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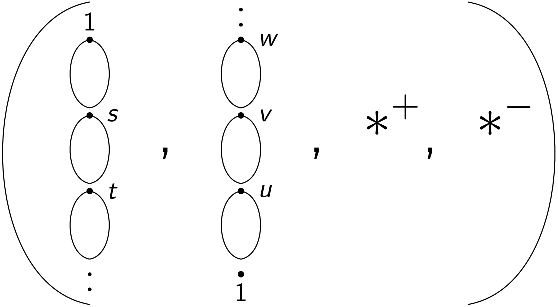

Let us call such class G_D and D the variety it generates.

A categorical equivalence for B

Consider the algebraic categories associated to B and D. We would like to show that this two categories are equivalent.

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Let $\mathcal{G} : \mathsf{B} \to \mathsf{D}$, then given $\mathbf{A} \in \mathsf{B}$,

$$\mathcal{G}(\mathbf{A}) = (\mathbf{A}^-, \mathbf{A}^+, *^+, *^-)$$

where $*^- : \mathbf{A}^+ \to \mathbf{A}^-$ and $*^+ : \mathbf{A}^- \to \mathbf{A}^+$ and given $x \in A^+$, $*^-(x) = x \to_A 1$ while if $x \in A^-$, $*^+(x) = x \to_A 1$.

A categorical equivalence for B

For the converse, let $\mathcal{F} : \mathsf{D} \to \mathsf{B}$. Then let $(\mathbf{A}, \mathbf{B}, *^+, *^-) \in \mathsf{D}$, then $\mathcal{F}((\mathbf{A}, \mathbf{B}, *^+, *^-))$ is the algebra having as domain:

$$\mathcal{F}((A, B, *^+, *^-)) = \{\langle a, b\rangle \in A \times B \mid \langle a, b\rangle = \pi(\langle a, b\rangle)\}$$

where $\pi(\langle a, b\rangle) = \langle (a \to_A b^{*^-}) \to_A a, (a \to_A b^{*^-})^{*^+} \wedge_B b\rangle$ and operations:

$$\begin{aligned}
\langle a, b\rangle \wedge \langle c, d\rangle &= \langle a \wedge c, b \wedge d\rangle \\
\langle s, b\rangle \vee \langle c, d\rangle &= \langle a \vee c, b \vee d\rangle \\
\langle a, b\rangle \cdot \langle c, d\rangle &= \pi(\langle ac, bd\rangle) \\
\langle a, b\rangle \to \langle c, d\rangle &= \pi(\langle \alpha, \beta\rangle)
\end{aligned}$$

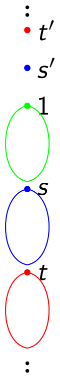
where $\alpha = (a \to_A c) \wedge_A ((b \to_B d)^{*^-}) \to_A b^{*^-})$ and $\beta = a^{*^+} \cdot_B (b \to_B d)$

A categorical equivalence for B

It can be shown that the functors $\mathcal{G} : \mathsf{B} \to \mathsf{D}$ and $\mathcal{F} : \mathsf{B} \to \mathsf{D}$ forms a categorical equivalence, giving that the algebraic category D and the algebraic category B are equivalent.

A categorical equivalence for B

Moreover, observe that any algebra $\mathbf{A}$ in the subvariety of B generated by algebras of this form



can be reconstructed starting only form the nuclear algebra $(\mathbf{A}^-, \gamma)$. Moreover, if the negative bubbles are idempotent chains, we get back Galatos and Raftery categorical equivalence.

Thanks for your attention