Arboreal Categories from Shapes

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Joint work with Tomáš Jakl and Luca Reggio

- Comonadic semantics attempts to associate to a logic $\mathcal{L}$ an adjunction

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[l, bend left, "M"] \end{tikzcd}$$

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- Usually arboreal adjunctions arise as comonadic adjunctions for "handcrafted" comonads, e.g.

Comonad	**Logic**	**Fragment**	**Reference**
Pebble comonad	FO	number of variables	[ADW17]
EF comonad	FO	quantifier rank	[AS21]
Guarded comonad	FO	guarded-quantifier depth	[AM21]
Modal comonad	BML	modal depth	[AS21]
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- In this talk, we reverse the usual script: we construct arboreal categories on the semantic side and observe that they fit naturally in arboreal adjunctions.

Definition

Let $\mathcal{S}$ be any category, whose objects we call **shapes**. The category $\mathcal{S}$-Tree is defined as follows.

- A **shape tree** (T, t) is a rooted tree T where each non-root node u is labelled by a shape $t(u) \in \mathcal{S}$.

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Example

$$
\begin{tikzcd}
 & u_2 \arrow[r] \arrow[drrr, bend left, mapsto] & u_3 \arrow[drrr, bend left, mapsto] & & & \\
u_0 \arrow[r] \arrow[ur] \arrow[rrr, bend right, mapsto] & u_1 \arrow[rrr, bend right, mapsto] & & v_0 \arrow[r] & v_1 \arrow[r] & v_2
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& t(u_2) \arrow[r] \arrow[drrr, bend left, "\gamma_{u_2}"] & t(u_3) \arrow[drrr, bend left, "\gamma_{u_3}"] & & & \\
\bullet \arrow[ur] \arrow[r] & t(u_1) \arrow[rrr, bend right, "\gamma_{u_1}"'] & & \bullet \arrow[r] & q(v_1) \arrow[r] & q(v_2)
\end{tikzcd}
$$

- We equip $\mathcal{S}$-Tree with an orthogonal factorisation system $(\mathcal{Q}, \mathcal{M})$ given by
  - $\mathcal{Q} =$ **surjective maps** ((f, γ) with f surjective)
  - $\mathcal{M} =$ **iso-injective maps** (f injective and γ_u iso $\forall u$).

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- To obtain a *proper* factorisation system, we require that **all morphisms in $\mathcal{S}$ are epis.**

Theorem

Let $\mathcal{S}$ be a category whose morphisms are all epimorphisms (and isomorphism classes are sets). Then $\mathcal{S}$-Tree equipped with $(\mathcal{Q}, \mathcal{M})$ is arboreal.
A path $(P, p) = [S_1, \ldots, S_\ell]$ is a linear shape tree.

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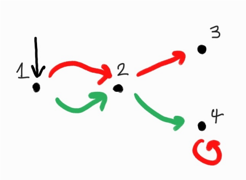
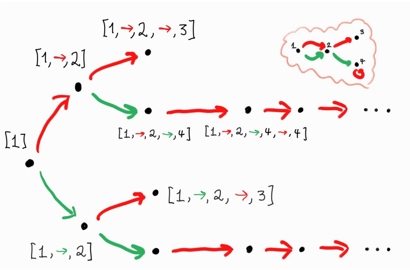
Intuition: we are generalising the game comonad $\mathbb{M}$ for Basic Modal Logic, based on the well-known unravelling construction.

(G, v_0) $\mapsto$ $\mathbb{M}(G, v_0)$





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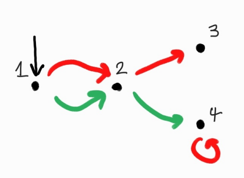

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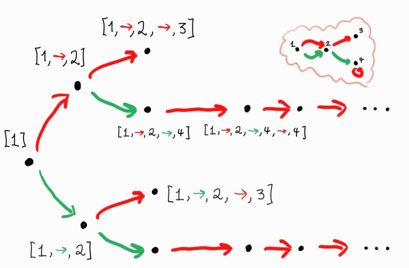

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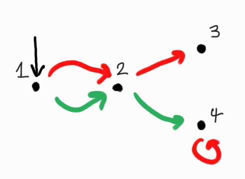

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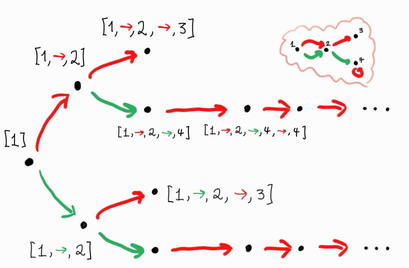
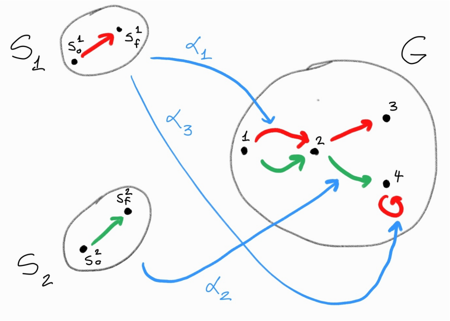


- A point in $\mathbb{M}(G, v_0)$ is a list e.g. $[1, {\color{red}\rightarrow}, 2, {\color{green}\rightarrow}, 4, {\color{red}\rightarrow}, 4]$ indicating an exploration history in G from v_0.
- *The color of the edge leading to the point is the last color inside the list.*

Now reinterpret a path as a sequence of morphisms or "probes" from **bi-pointed** shapes S (with source s_0 and target s_f):

$$[1, \rightarrow, 2, \rightarrow, 4, \rightarrow, 4] \approx [\alpha_1, \alpha_2, \alpha_3]$$



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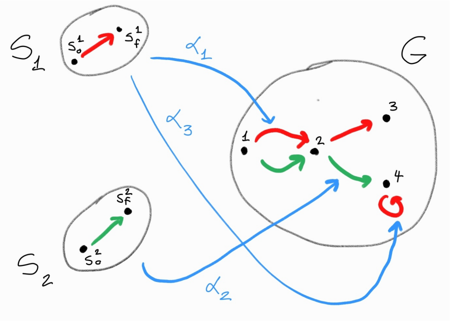
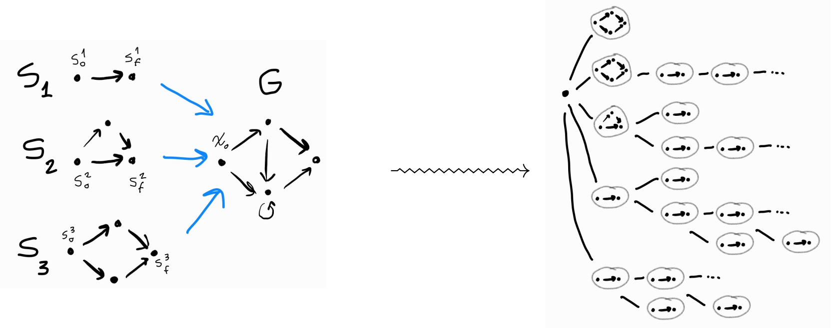
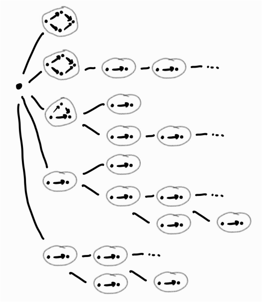


- $\alpha_1(s_0^1) = x_0$
- Consecutive endpoints match: $\alpha_i(s_f^i) = \alpha_{i+1}(s_0^{i+1})$.
- The resulting lists $\vec{\alpha} = [\alpha_1, \ldots, \alpha_\ell]$ form a tree by prefix order.
- *The list $\vec{\alpha}$ is "coloured" with the last shape,* dom(α_ℓ)*.*

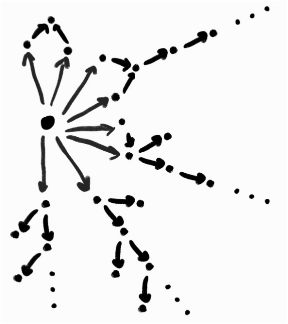
From this new perspective, we can probe a graph with more complex shapes. However the result is not a graph, but a shape tree, e.g.



To recover a graph ("the unravelling") from the shape tree of explorations, take a big colimit:



$\rightsquigarrow$



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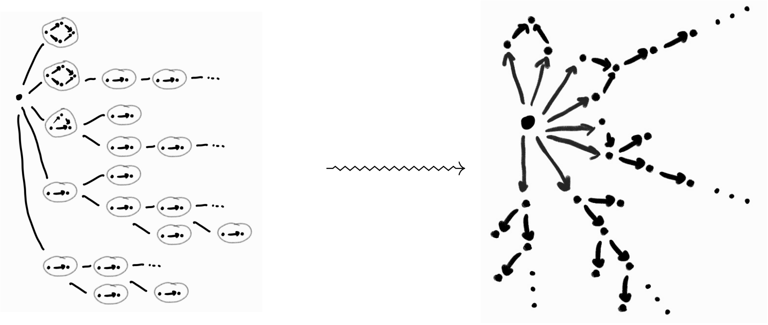

These two steps correspond to the right and left adjoints of our arboreal adjunction.

Let $\mathcal{E}$ be cocomplete. Fix an object $* \in \mathcal{E}$ (arbitrary!)

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Definition

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Assumption

$\mathcal{S}$ is a subcategory of $\mathcal{E}_{**}$ (possibly non-full).

The left adjoint: a path $(P, p) = [S_1, \ldots, S_\ell]$ is realised as the colimit

$$\begin{tikzcd}
* \arrow[dr, "s_0^1"] & & * \arrow[dl, "s_f^1"'] \arrow[dr, "s_0^2"] & & \cdots & & * \arrow[dl, "s_f^{\ell-1}"'] \arrow[dr, "s_0^\ell"] & & * \arrow[dl, "s_f^\ell"'] \\
& S_1 & & S_2 & & S_{\ell-1} & & S_\ell &
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& S_1 \arrow[drrr, dashed] & & S_2 \arrow[dr, dashed] & & S_{\ell-1} \arrow[dl, dashed] & & S_\ell \arrow[dlll, dashed] & \\
& & & & M(P, p) & & & &
\end{tikzcd}$$

with basepoint x_0. Since paths are dense, this defines a functor

$$M \colon \mathcal{S}\text{-Tree} \to \mathcal{E}_*.$$

The right adjoint:

- Given $(X, x_0) \in \mathcal{E}_*$, let

$$R(X, x_0) := \{[\alpha_1, \ldots, \alpha_\ell] \mid \ell \geq 0, \alpha_i : S_i \to X, \\ \alpha_1 s_0 = x_0, \\ \alpha_{i+1} s_0 = \alpha_i s_f\}$$

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- Given $f : (X, x_0) \to (Y, y_0)$, let

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Rf : R(X, x_0) &\to R(Y, y_0) \\
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- This is great... Except it only works if $\mathcal{S}$ is a set (i.e. discrete and small).

Assumption/Definition

We assume that the forgetful functor $\mathcal{S} \to \mathcal{E}_{**} \to \mathcal{E}$ **admits a right multi-adjoint** [Die81]: for each $X \in \mathcal{E}$ each connected component of $\mathcal{S} \downarrow X$ has a terminal object (and there is a set's worth of c.c.s).

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Concretely, given $\alpha : S \to X$ there is a chosen **saturation**

$$\overline{\alpha} : \overline{S} \to X$$

that is terminal in the connected component of α. In particular, there exists a unique ζ_α as in the diagram:

$$\begin{tikzcd} S \arrow[rr, "\exists! \zeta_\alpha"] \arrow[dr, "\alpha"'] & & \overline{S} \arrow[dl, "\overline{\alpha}"] \\ & X & \end{tikzcd}$$

Call α **saturated** if $\alpha = \overline{\alpha}$.

Definition (Right adjoint $R : \mathcal{E}_* \to \mathcal{S}\text{-Tree}$)

- Given $(X, x_0) \in \mathcal{E}_*$, let

$$R(X, x_0) := \{[\alpha_1, \dots, \alpha_\ell] \mid \ell \geq 0, \alpha_i : S_i \to X, \\ \alpha_1 s_0 = x_0, \\ \alpha_{i+1} s_0 = \alpha_i s_f, \\ \alpha_i = \overline{\alpha}_i\}$$

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Definition (Right adjoint $R : \mathcal{E}_* \to \mathcal{S}$-Tree, contd.)

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and for each $\vec{\alpha} = [\alpha_1, \ldots, \alpha_\ell] \neq []$, let the morphism of shapes be

$$
\begin{tikzcd}
S_\ell \arrow[r, "(Rf)_\alpha := \zeta_{f \cdot \alpha}"] \arrow[d, "\alpha_\ell"'] & \overline{S}_\ell \arrow[d, "\overline{f \cdot \alpha_\ell}"] \\
X \arrow[r, "f"'] & Y
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Theorem

Let $\mathcal{S} \subseteq \mathcal{E}_{**}$ be a subcategory of epimorphisms such that

$$\mathcal{S} \to \mathcal{E}_{**} \to \mathcal{E}$$

admits a right multi-adjoint. Then the functors

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Theorem

If there is a forgetful functor $\mathcal{E} \to \mathbf{Set}$ preserving colimits and no shape $S \in \mathcal{S}$ is a singleton, then the adjunction is comonadic.

Definition

Let $\mathcal{L}_{\mathcal{S}}$ denote the language with syntax

$$\varphi \ ::= \top \ | \ \neg\varphi \ | \ \bigwedge_{i \in I} \varphi_i \ | \ \langle S \rangle \varphi \qquad\qquad (S \in \mathcal{S},\ I \in \mathrm{Set})$$

and semantics on objects $(X, x_0) \in \mathcal{E}_*$ given by

$$\begin{array}{ll} X, x_0 \models \top & \text{always} \\ X, x_0 \models \neg\varphi & \text{not } X, x_0 \models \varphi \\ X, x_0 \models \bigwedge_{i \in I} \varphi_i & X, x_0 \models \varphi_i \text{ for all } i \in I \\ X, x_0 \models \langle S \rangle \varphi & \text{iff there is some } \alpha : S \to X \text{ such that} \\ & \alpha(s_0) = x_0, \\ & X, \alpha(s_f) \models \varphi, \text{ and} \\ & \overline{\alpha} = \alpha. \end{array}$$

Theorem

The arboreal adjunction $\mathcal{E}_* \leftrightarrows \mathcal{S}$-Tree **captures** logical equivalence for $\mathcal{L}_{\mathcal{S}}$, i.e.

$$(X, x_0) \equiv_{\mathcal{L}_{\mathcal{S}}} (Y, y_0) \iff R(X, x_0) \rightleftharpoons R(Y, y_0) \text{ in } \mathcal{S}\text{-Tree.}$$

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Conjecture

The comonad $\mathbb{D} = MR$ on $\mathcal{E}_*$ has the **bisimilar companion property**: for all $(X, x_0) \in \mathcal{E}_*$, $(X, x_0) \rightleftharpoons \mathbb{D}(X, x_0)$.

Example: BML and extensions ($\mathcal{S}$ discrete)

1. Let $\sigma = \mathrm{PROP} \cup \mathrm{ACC}$ (unary and binary relations) and $\mathcal{S} \subseteq \mathrm{Struct}_{**}(\sigma)$ be discrete on objects:

$$\begin{array}{ccccc} S_p & & & S_a & \\ \bullet^p & (p \in \mathrm{PROP}) & & \underset{s_0}{\bullet} \xrightarrow{a} \underset{s_f}{\bullet} & (a \in \mathrm{ACC}) \\ s_0 = s_f & & & & \end{array}$$

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$$S_a \qquad \underset{s_0}{\bullet} \xrightarrow{a} \underset{s_f}{\bullet} \quad (a \in \mathrm{ACC})$$

   Then the comonad $\mathbb{D}$ captures BML, where prop. variables are encoded as modalities of arity 0. *It is not isomorphic to the modal comonad $\mathbb{M}$.*

2. We can extend BML with modalities such as loop $\circlearrowleft$, backwards $\leftarrow$ and "sibling" by adding the shapes:

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd} \quad s_0 = s_f \qquad\qquad \underset{s_0}{\bullet} \longleftarrow \underset{s_f}{\bullet} \qquad\qquad \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \underset{s_0}{\bullet} & & \underset{s_f}{\bullet} \end{tikzcd}$$

Example: ML + CQs ($\mathcal{S}$ discrete)

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- **Conjecture:** we can "add tests" to the logic $\mathcal{L}_{\mathcal{S}}$ by a suitable modification of $\mathcal{S}$-Tree ("unravelling along shapes without fixed targets").

Example: Logics of embeddings ($\mathcal{S}$ *non-discrete*)

- Let $\mathcal{S}_0 \subseteq \mathcal{E}_{**}$ be a small subcategory of epimorphisms and let $\mathcal{S}$ be the upwards closure of $\mathcal{S}_0$ under epimorphisms.

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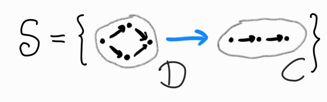
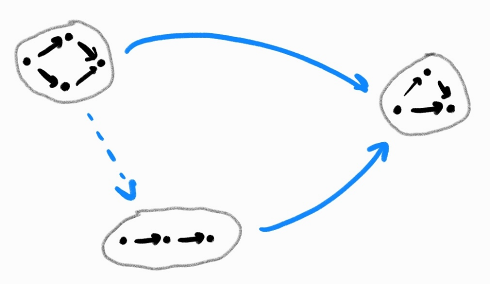
- As a sub-example,



satisfies the axioms (source/target depicted on the left/right).

Example: Logics of embeddings ($\mathcal{S}$ *non-discrete*) (contd.)

- Now let (X, x_0) be the directed triangle on the right.



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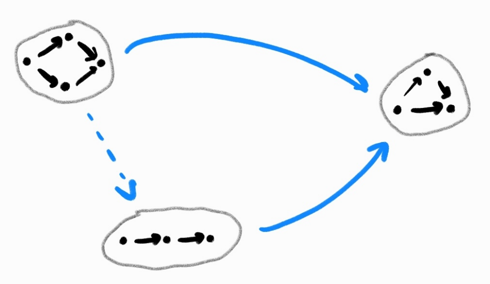

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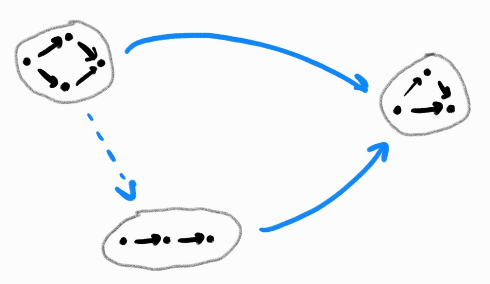

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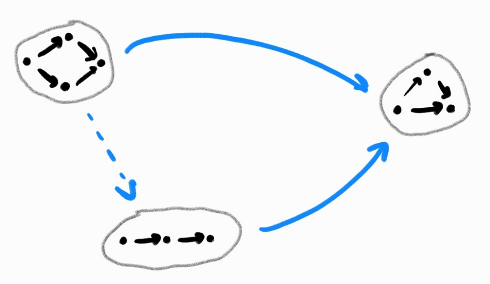

- The unique map $D \to X$ is not terminal in its own c.c. of $\mathcal{S} \downarrow X$, thus it is not saturated.
- Hence $(X, x_0) \not\models \langle D \rangle \top$.
- If we had not included the epimorphism $D \twoheadrightarrow C$ in $\mathcal{S}$, then we would have $(X, x_0) \models \langle D \rangle \top$.

Thank you!

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