DELIMITED CONTROL AND MODAL INTUITIONISTIC LOGIC

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OUTLINE

LAMBDA CALCULUS AND IQC

- IQC = Intuitionistic predicate logic, as presented by standard Gentzen-style natural deduction.
- Lambda terms = annotations for natural deduction derivations
- Inference rules = term constructors
- Dynamic behavior = reduction rules = proof normalization

$$\frac{\Gamma \vdash p : A \to B \qquad \Gamma \vdash q : A}{\Gamma \vdash pq : B}(\to E) \qquad \frac{\Gamma, a : A \vdash p : B}{\Gamma \vdash \lambda a.p : A \to B}(\to I)$$

$$\frac{\Gamma \vdash p : \forall x\, P(x)}{\Gamma \vdash pt : P(x := t)}(\forall E) \qquad \frac{\Gamma \vdash p : P(x) \qquad x \text{ not free in } \Gamma}{\Gamma \vdash \lambda x.p : \forall x\, P(x)}(\forall I)$$

$$\frac{\Gamma \vdash p : \bot}{\Gamma \vdash \varepsilon(p) : A}(\bot E) \qquad (\lambda x.p)t \mapsto p[x := t]$$

- The calculus of proof terms for IQC is *strongly normalizing*, i.e.
  - Every proof term reduces to a ‘value’
  - Every proof can be transformed to a normal form
- It follows that IQC has the *disjunction and existence properties*
  - If $\vdash A \vee B$, then either $\vdash A$ or $\vdash B$
  - If $\vdash \exists x\, P(x)$, then $\vdash P(x := t)$, for some closed term t.
- One of many ways to justify “intuitionistic”.

A COMPUTATIONAL INTERPRETATION OF CLASSICAL LOGIC?

- Control operators are a functional way to give programs/lambda terms the facility to manipulate the control flow of the program
  - similar to `try`, `catch`, `throw`, `goto`, `break`, etc.
- Griffin (1989) noticed that popular control flow operators (implemented as axioms/built-ins) type various forms of excluded middle:
  - Scheme: `call/cc` (*call-with-current-continuation*)
  - Felleisen's $\mathcal{C}$

$$\underbrace{\texttt{call/cc} : \neg\neg A}_{\text{double negation}} \qquad \equiv \qquad \underbrace{\mathcal{C} : (\neg A \to A)}_{\text{Pierce } \checkmark}$$

Can also implement at the syntactic level, via a new term constructor:

$$\frac{\Gamma, k : A \to \bot \vdash p : A}{\Gamma \vdash \&k.p : A} (\& \text{ “shift”})$$

Computational interpretation: control flow manipulation

Writing $\&k.(\dots)$ in a lambda term captures the *current continuation* and binds it to the variable k:

$$\lambda a.\lambda b.\dots \boxed{\&k.(\dots)} \dots \qquad k = \underbrace{\lambda x.\left(\lambda a.\lambda b.\dots \boxed{x} \dots\right)}_{\text{“continutation” of the boxed sub-term}}$$

k may be called inside the body with a value x; the computation is “aborted” and continues as if the whole boxed expression evaluated to x. This intent is captured by new reduction rules / operational semantics*:

$$E[\ \&k.p\ ] \mapsto p[k := \lambda x.E[x]] \qquad \text{(but more carefully)}$$

A "construction" of excluded middle:

$$\underset{\substack{\uparrow \\ (A\vee\neg A)\to\bot}}{\&k}.\quad \iota_2(\ \lambda \underset{\substack{\uparrow \\ A}}{a}.\ \ \underbrace{k(\iota_1 a)}_{\bot}\ )\ :\ A \vee \neg A$$

- This term chooses right ($\neg A$), and provides a "construction" of $A \to \bot$...
- But if it's ever invoked (by the rest of the proof/program) with a proof of A, it "changes its mind" and chooses the left option; the surrounding program could "jump" to a different branch during evaluation.
- This lambda calculus *is* normalizing, but terms may reduce to different results depending on the context (and inspecting the types doesn't help) – debatable whether a term represents a "construction".
- Further on this topic: Fully symmetrized lambda calculus using a term/co-term routine/co-routine duality to capture the full two-sided classical sequent calculus

OUTLINE

1. Lambda calculus and control operators
2. Delimited control
3. Multi-prompt delimited control
4. Disjunction and existence

DELIMITED CONTROL

$$\frac{\Gamma, k : A \to \bot \Vdash_\bullet p : A}{\Gamma \Vdash_\bullet \&k.p : A}(\&\text{ “shift”}) \qquad \frac{\Gamma \Vdash_\bullet p : \bot}{\Gamma \Vdash \#p : \bot}(\#\text{ “reset”})$$

- Turnstile ($\Vdash$ for this system) can carry a “decoration”
- Decoration freely available on axioms; ignored by intuitionistic rules
- The Pierce rule is available <u>only</u> in a decorated context; the decoration can only be removed by reset, when the goal is $\bot$

$$\lambda a.\lambda b.\ldots \quad \underset{\substack{\uparrow \\ \text{goal:}\bot}}{\#} \quad \lambda c.\ldots \boxed{\&k.(\ldots)}\ldots \qquad k = \lambda x.\ \left(\lambda c.\ldots\boxed{x}\ldots\right)$$

The current <u>delimited</u> continutation (up to the nearest #) is captured and bound to *k*. Corresponding dynamic behavior/reduction rules.

We no longer have full classical power. But it was noticed by Ilik (2012) that delimited control provides strictly more than just intuitionistic power:

$$\lambda a. \leftarrow \forall x \, \neg\neg Px$$

$$\lambda b. \leftarrow \neg\forall x \, Px$$

$$\underset{\substack{\uparrow \\ \text{goal:}\bot}}{\#} \;\; b(\; \lambda x. \;\; \&\underset{\substack{\uparrow \\ \neg Px}}{k}. \;\; \varepsilon(\; \underbrace{axk}_{\bot} \;)\;) \qquad k = b(\lambda x.\boxed{-})$$

$$: \forall x \, \neg\neg Px \to \neg\neg\forall x \, Px$$

This is the Kuroda principle K or Double Negation Shift DNS.

BACK TO LOGIC

Smetanich (1960) considered the following extension of intuitionistic logic.
Extend the langauge by a constant symbol Υ.
Add the following axioms; refer to the system as IQCΥ.

- $\Upsilon \to ((\neg p \to p) \to p)$ or: $\Upsilon \to (\neg\neg p \to p)$
- $\neg\Upsilon \to \Upsilon$ or: $\neg\neg\Upsilon$

Observations:

- Kripke semantics: Υ should hold at maximal points (i.e., “classical worlds”), and such a world is consistent.
- In a complete Heyting algebra, $\Upsilon = \bigwedge \{x \in H : \neg\neg x = 1\}$ (“strongest consistent proposition”)
- Conservative over IPC (intuitionistic *propositional* logic) using FMP.

IQCϒ is not conservative over IQC:

$$
\begin{array}{rl}
1 & \forall x \, \neg\neg P(x) \\
2 & \quad \neg\forall x \, P(x) \\
3 & \quad\quad \Upsilon \\
4 & \quad\quad \neg\neg P(a) \equiv \neg P(a) \to P(a) \\
5 & \quad\quad P(a) \\
6 & \quad\quad \forall x \, P(x) \\
7 & \quad\quad \bot \\
8 & \quad \neg\Upsilon \\
9 & \quad \Upsilon, \text{ thus } \bot \\
10 & \neg\neg\forall x \, P(x)
\end{array}
$$

- $\vdash$ = derivability in IQC
- $\Vdash$ = derivability in IQC with rules shift, reset
- $\vdash_S$ = derivability in an axiomatic extension S of IQC.

Theorem (M.)

Let A be an intuitionistic formula.

- $\Gamma \Vdash_{\bullet} A \quad \Leftrightarrow \quad \Gamma, \Upsilon \vdash_{\mathrm{IQC}\Upsilon} A$
- $\Gamma \Vdash A \quad \Leftrightarrow \quad \Gamma \vdash_{\mathrm{IQC}\Upsilon} A$

i.e., $\Vdash$ is a natural deduction system/lambda calculus for the intuitionistic fragment of IQCΥ

- Proof of ($\Rightarrow$): example on last slide shows how to use Υ to emulate shift and reset.

- Proof of $(\Leftarrow)$ comes down to the following fact: IQC $+$ K is the intuitionistic fragment of IQCΥ.
- i.e., K accounts for the whole of the residual effects of Υ.
- Known that IQC $+$ K is complete with respect to predicate Kripke frames with maximal points (*this phenomenon is rare*)
- Falsify A on such a model; define Υ on the maximal points to obtain a model of IQCΥ falsifying A.
- Kripke completness also be used to show that IQC $+$ K has the disjunction and existence properties.– i.e. this calculus is “constructive”.
  - Ilik gives a proof based on a multi-tiered CPS translation/embedding of the lambda terms.

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MULTI-PROMPT DELIMITED CONTROL

$$\frac{\Gamma, k : B \to A \Vdash_{\Delta} p : B \qquad \alpha : A \in \Delta}{\Gamma \Vdash_{\Delta} \&_{\alpha} k.p : B}(\&\ \text{“shift”}) \qquad \frac{\Gamma \Vdash_{\alpha:A,\Delta} p : A}{\Gamma \Vdash_{\Delta} \#_{\alpha} p : A}(\#\ \text{“reset”})$$

- Here the turnstile carries a context Δ of *prompt variables* $\alpha, \beta, \ldots$ which are also typed by formulas.
- Pierce “relative to A” is available in an A-context (when a prompt variable $\alpha : A$ is in the context Δ).
- An A-context can be escaped when the goal is A.
- The choice of “Pierce version” is essential here!

Observation:

$$\lambda a. \leftarrow \forall x((Px \to \forall y\, Py) \to \forall y\, Py)$$

$$\underset{\substack{\uparrow \\ \forall x\, Px \\ \text{(goal)}}}{\#_\alpha}\ \lambda x.\ \underset{\substack{\uparrow \\ Px \to \forall x\, Px}}{\&_\alpha k.}\ \underbrace{axkx}_{Px} \qquad k = \lambda x.\boxed{-}$$

$$: \forall x\,((Px \to \forall y\, Py) \to \forall y\, Py) \to \forall x\, Px$$

- This is the *Casari schema* Cas, a.k.a. the *relativized Kuroda principle*.
- Though much more obtuse than DNS, this is strictly stronger than DNS. It shows up in interesting places:
  - Necessary to: extend the *provability interpretation* of intuitionistic propositional logic to the mondaic first-order setting (Bezhanishvili, Brantley, Ilin), extend the characterization of local finiteness in S4-algebras to the monadic setting (M.)

In the 1970s Kuznetsov and Muravitsky arrived at the intuitionistic modal logic KM.

Extend the language by a unary modal operator $\Box$.

Add the following axioms

- $p \to \Box p$ (*completeness*)
- $\Box p \to ((q \to p) \to q) \to q$ (*Cantor–Bendixon axiom*)
- $(\Box p \to p) \to p$ (*strong Löb axiom*) or: $(\Box p \to p) \to \Box p$

Has a provability interpretation (= embedding into GL, the provability logic of Peano arithmetic) extending the one for IPC.

In this interpretation, propositions stand for *true and provable* arithmetic sentences. Adding $\Box$ “returns” the provability predicate to intuitionistic logic, granting the ability make assertions solely about provability.

This grants no new power at the propositional level (conservative over IPC).

Esakia observed that QKM is not conservative over IQC:

$$
\begin{array}{r|l}
1 & \forall x\,((Px \to \forall y\,Py) \to \forall y\,Py) \\
\hline
2 & \quad \begin{array}{|l} \Box \forall x\, Px \\ \hline (Pa \to \forall y\,Py) \to \forall y\,Py \\ (Pa \to \forall y\,Py) \to Pa \\ Pa \\ \forall x\,Px \end{array} \\
7 & \Box \forall x\,Px \to \forall x\,Px \\
8 & \forall x\,Px
\end{array}
$$

1	$\forall x\,((Px \to \forall y\,Py) \to \forall y\,Py)$
2	$\quad \Box \forall x\, Px$
3	$\quad (Pa \to \forall y\,Py) \to \forall y\,Py$
4	$\quad (Pa \to \forall y\,Py) \to Pa$
5	$\quad Pa$
6	$\quad \forall x\,Px$
7	$\Box \forall x\,Px \to \forall x\,Px$
8	$\forall x\,Px$

Observation: instantiate $p = \bot$ in the axioms:

- $\bot \to \Box\bot$
- $\Box\bot \to ((\neg p \to p) \to p) \equiv (\neg\neg p \to p)$
- $\neg\neg\Box\bot$

In fact, Υ is definable as $\Box\bot$ ($\Box\bot$ is the "strongest consistent proposition").

One can also look at the axioms for both systems in the following framing:

Nucleus j	element with $j(-) = 1$	that implies *j-stability*
$n(x) = \neg x \to x$	$\neg\Upsilon \to \Upsilon$	$\Upsilon \to ((\neg x \to x) \to x)$
$s_A(x) = (x \to A) \to x$	$(\Box A \to A) \to \Box A$	$\Box A \to ((x \to A) \to x) \to x$

Another observation: DNS $\equiv$ "n-shift": $\forall x\, n(Px) \to n(\forall x\, Px)$

Generalizes to: Cas $\equiv$ "s_A-shift": $\forall x\, s_A Px \to s_A \forall x\, Px$ for all A

Conjecture

Let B be an intuitionistic formula.

- $\Gamma \Vdash_{\Delta} B \quad \Leftrightarrow \quad \Gamma, \{\Box A : A \in \Delta\} \vdash_{\mathsf{QKM}} B$
- $\Gamma \Vdash B \quad \Leftrightarrow \quad \Gamma \vdash_{\mathsf{QKM}} B$

i.e., multi-prompt $\Vdash$ is a natural deduction system/lambda calculus for the intuitionistic fragment of QKM

Proof of $(\Rightarrow)$: Example on previous slide shows how to emulate relativized shift/reset with $\Box$.

Proof of ($\Leftarrow$) would rely on the corresponding claim that
IQC + Cas is the intuitionistic fragment of QKM.

- Much more difficult, since IQC + Cas is certainly not Kripke complete – Cas even on monadic Heyting algebras is known to be non-canonical.
- Constructions embedding a Heyting algebra into a KM-algebra preserving the logic are technical, making the construction difficult to extend to predicate semantics.
- It's known (Esakia, Jibladze, Pataraia, 2000) that, for a topos, which always has a propositional operator $\Box : \Omega \to \Omega$ on the complete Heyting algebra Ω satisfying the first two KM axioms, the validity of the strong Löb axiom is equivalent to the validity of the Casari schema.

OUTLINE

- The disjunction and existence properties are a compelling characterization of the “constructive” character of a logic.
- Syntactic approach: e.g., strong normalization of proofs.
  - Does not appear to be done for the generalized multi-prompt delimited control calculus.
- Semantic approach: “Gluing” type constructions
  - At the propositional level: Show axioms are preserved under the operation that adds a new top element to a Heyting algebra.
  - e.g., KM has the disjunction property by this construction.
  - Generalized to the predicate logic case: Artin gluing and the *Freyd cover* on topoi and hyperdoctrines.

HYPERDOCTRINE SALES PITCH

Hyperdoctrines provide a direct analogue of algebraic semantics for predicate logic, and can be thought of as fibered/indexed algebras. e.g., a doctrine over Heyting algebras is:

$$H : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

$$\begin{tikzcd} P(K \times I) \arrow[d, bend right, "\exists"'] \arrow[d, bend left, "\forall"] \\ P(I) \arrow[u, "\pi^*"'] \end{tikzcd}$$

where each projection has left and right adjoints satisfying some additonal conditions (Beck–Chevalley; compability of quantifiers with substitution).

- General completeness lifting: Predicate extension of a propositional logic has adequate semantics given by doctrines over its algebras (from the construction of a “Lindenbaum–Tarski” doctrine).

Hyperdoctrine semantics and gluing constructions have been employed by Shirasu (1995) to demonstrate the disjunction and existence properties for predicate substructural logics, and Pitts (1991) uses the technique as an alternative way to demonstrate DP and EP for IQC. Roughly:

$$
\begin{tikzcd}
\mathbf{C}^{op} \arrow[dd, "H"'] \arrow[dr, bend left, "\Gamma"] & \\
{} \arrow[r, Rightarrow, "\gamma"] & \mathbf{Set}^{op} \arrow[dl, bend left, "\mathcal{P}"] \\
\mathbf{HA} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}^{op} \arrow[d, "\widehat{H}"] \\
\mathbf{HA}
\end{tikzcd}
\qquad
\widehat{H}(I) = \mathrm{Gl}(\ H(I) \xrightarrow{\gamma_I} \mathcal{P}\Gamma(I)\ )
$$

where Γ is the global sections functor, $\mathcal{P}$ is the power-set hyperdoctrine. The new doctrine $\widehat{H}$ is formed by the fiberwise “gluing” along γ_I.

- Can we glue KM-algebras instead of Heyting algebras?
- What kind of (predicate) axioms are preserved under this construction?

What is $\mathrm{Gl}(\mathfrak{A} \xrightarrow{f} \mathfrak{B})$ when $A, B \in \mathbf{HA}$? It is the Heyting algebra with carrier

$$\{(b,a) \in \mathfrak{B} \times \mathfrak{A} : b \leq f(a)\}$$

with order and operations defined pointwise, except:

$$(b,a) \to (b',a') = ((b \to b') \wedge f(a \to a'),\ a \to a')$$

The "correct" extension to KM-algebras (informed by the topological semantics of $\Box$ as the Cantor–Bendixon *co-derivative*) is:

$$\Box(b,a) = (\Box b \wedge f(a),\ \Box a)$$

under which the glued algebra will again be a KM-algebra, and hence:

Theorem (M.)

QKM *has the disjunction and existence properties.*

As one might hope, it turns out the axiom Cas, i.e.

$$\forall((p \to \pi^* \forall p) \to \pi^* \forall p) \leq \forall p \qquad p \in H(K \times I)$$

is preserved under the gluing construction.

Theorem (M.)
IQC + Cas *has the disjunction and existence properties.*

- Classes of formulae preserved under gluing have been previously studied by analyzing the preservation properties of the Freyd cover on topoi – they include, e.g., DNS but not Cas – can these classes be generalized?

Thanks!