Lax comma categories: descent and exponentiability

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Motivation

- Develop a higher-dimensional counterpart of *categorical Galois theory*:

  Clementino, M.M., Lucatelli Nunes, F.: Lax comma 2-categories and admissible 2-functors. *Theory Appl. Categ.*, 2024.

- To do so, we should understand the *descent theory* and *exponentiability* in these categories.

  Ord-enriched setting particularly convenient.

- Potential application to $\mathsf{OrdCH} \to \mathsf{Priest}$?

Lax comma categories

Let $\mathbb{A}$ be a 2-category, X in $\mathbb{A}$, $\mathbb{S} \subseteq \mathbb{A}$ a full sub-2-category.

The lax comma (2-)category $\mathbb{S} \Downarrow X$ has

- objects: morphisms $f\colon S \to X$ of $\mathbb{A}$ with S in $\mathbb{S}$,
- morphisms $f \to g$: a morphism h and a 2-cell θ in $\mathbb{A}$ as below:

$$\begin{tikzcd} S \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & T \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- 2-cells...

Examples

- $\mathsf{Set} \Downarrow \mathcal{X} \simeq \mathsf{Fam}(\mathcal{X})$.

  $\mathbb{A} = \mathsf{CAT}$, $\mathbb{S} = \mathsf{Set}$, and $\mathcal{X}$ a category.

- $\mathsf{Ord} \Downarrow X =$ category of *ordered families*.

  $\mathbb{A} = \mathbb{S} = \mathsf{Ord}$, X an ordered set.

- $\mathsf{Cat} \Downarrow \mathcal{X} =$ category of *diagrams in* $\mathcal{X}$.

  $\mathbb{A} = \mathsf{CAT}$, $\mathbb{S} = \mathsf{Cat}$, $\mathcal{X}$ a category.

- $\mathsf{Top} \Downarrow X$.

  Clementino, M.M., Hofmann, D., R.P.: Topological lax comma categories. *Order*, 2026.
  Main focus of this presentation.

Grothendieck construction

The lax comma category $\mathbb{S} \Downarrow X$ is equivalent to the total category of the *Grothendieck construction* of the (strict) 2-functor

$$\begin{aligned} \mathbb{A}(i(-), X) \colon \mathbb{S}^{\mathsf{op}} &\longrightarrow \mathsf{CAT} \\ S &\longmapsto \mathbb{A}(S, X) \\ f \colon S \to T &\longmapsto - \cdot f \colon \mathbb{A}(T, X) \to \mathbb{A}(S, X) \end{aligned}$$

that is, we have a fibration $\mathbb{S} \Downarrow X \to \mathbb{S}$.

Properties of lax comma categories

Intuition: properties of $\mathbb{S} \Downarrow X$ are an "external" counterpart of the properties of X (internally to $\mathbb{A}$).

- Limits and colimits.
- Distributivity of colimits over limits.
- Extensivity.
- Topologicity.
- Descent theory (descent & effective descent morphisms).
- Exponentiable objects.

Basic properties, observations

Let X be a topological space.

- If X is $T1$, then $\mathsf{Top} \Downarrow X = \mathsf{Top} \downarrow X$.
- The $T0$-reflection $\eta_X : X \to \pi_0(X)$ induces $\mathsf{Top} \Downarrow X \simeq \mathsf{Top} \Downarrow \pi_0(X)$.
- $\mathsf{Top} \Downarrow X$ has equalisers.
- $\mathsf{Top} \Downarrow X$ is infinitary extensive.

Let X be a T0-space.

We say X is a *topological* $\wedge$-semilattice if the convergence order has continuous finite meets.

We say X is a *topological* $\bigwedge$-semilattice if the convergence order has all continuous meets.

Lemma (R.E. Hoffman 1979)

The following are equivalent:

- X is sober.
- X has topological codirected meets.

Let X be a T0-space.

Theorem (D. Hofmann, 2014)

The following are equivalent:

- X is sober and a topological $\wedge$-semilattice.
- X is a topological $\bigwedge$-semilattice.
- The convergence order of X is a complete lattice, and ultrafilters in X have smallest points of convergence.

Corollary (M.M. Clementino, D. Hofmann, R.P., 2026)

The following are equivalent:

- X is a topological $\bigwedge$-semilattice.
- $\mathsf{Top} \Downarrow X$ has all small limits, and $\mathsf{Top} \Downarrow X \to \mathsf{Top}$ preserves them.

Exponentiability

We say X is a topological Heyting $\wedge$-semilattice if $x \wedge - \colon X \to X$ has a right adjoint $x \Rightarrow -$ in Top.

Theorem (M.M. Clementino, D. Hofmann, R.P., 2026)

If X is a topological Heyting $\bigwedge$-semilattice, then following are equivalent:

- $(A, \alpha \colon A \to X)$ is exponentiable in $\mathsf{Top} \Downarrow X$.
- A is core-compact.

For (B, β) in $\mathsf{Top} \Downarrow X$,

$$(B, \beta)^{(A,\alpha)} = \Big(B^A, h \mapsto \bigwedge_{a \in A} \alpha(a) \Rightarrow \beta(h(a))\Big)$$

Effective descent morphisms

Intuition: Descent = Decomposition + Gluing structure + Compatibility

Let $\mathcal{C}$ be a category with pullbacks.

$$\begin{tikzcd} \mathcal{C} \downarrow y \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \downarrow x \arrow[l, bend right, "p_!"'] \end{tikzcd}$$

We say p is an *effective descent morphism* (descent morphism) if p^* is monadic (premonadic).

Effective descent morphisms

Theorem

If a fibration $F\colon \mathcal{E} \to \mathcal{B}$ preserves pullbacks, then it preserves descent and effective descent morphisms.

Lucatelli Nunes, F., R.P.: Functors preserving effective descent morphisms. *arXiv preprint*, 2024.

Corollary

If X is a topological $\wedge$-semilattice, then $\mathsf{Top} \Downarrow X \to \mathsf{Top}$ preserves effective descent morphisms.

Behind the characterisation

$$\begin{tikzcd}
\mathsf{Top} \Downarrow X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathsf{Top} \times_{\mathsf{Set}} \mathsf{Fam}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{2-p.b.}"] & \mathsf{Fam}(X) \arrow[d] \\
& \mathsf{Top} \arrow[r] & \mathsf{Set}
\end{tikzcd}$$

R.P.: On effective descent V-functors and familial descent morphisms. *J. Pure Appl. Algebra*, 2024.

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Thank you!