There are only denumerably many locally tabular bi-intermediate logics of trees and of co-trees

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Bi-intuitionistic propositional calculus

Bi-intuitionistic propositional calculus

bi-IPC is the natural symmetric extension of **IPC**, and can be obtained by enlarging the language of **IPC** with the co-implication $\leftarrow$.

New axioms

- $p \to \big(q \vee (p \leftarrow q)\big)$,
- $\neg(p \leftarrow q) \to (p \to q)$,
- etc.

New inference rule

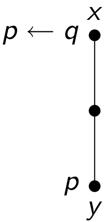
- Double negation: $\dfrac{\varphi}{\neg\sim\varphi}$

- The negation is defined by $\neg\varphi := \varphi \to \bot$.
- The co-negation is defined by $\sim\varphi := \top \leftarrow \varphi$.

The Kripke semantics of **bi-IPC**

- The Kripke semantics of **bi-IPC** provides a transparent interpretation of co-implication:

$$\mathfrak{M}, x \models p \leftarrow q \iff \exists\, y \leq x \; (\mathfrak{M}, y \models p \text{ and } \mathfrak{M}, y \not\models q).$$



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- Notice the dual behaviour to the intuitionistic implication:

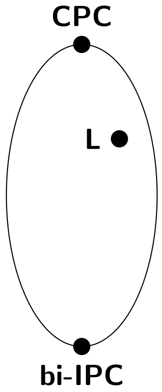
$$\mathfrak{M}, x \models p \rightarrow q \iff \forall y \geq x \,(\mathfrak{M}, y \not\models p \text{ or } \mathfrak{M}, y \models q).$$

Bi-intermediate logics

- We have **bi-IPC** $\subsetneq$ **CPC** because, e.g.,

$$\mathbf{CPC} \vdash (p \leftarrow q) \leftrightarrow (p \wedge \neg q) \quad \text{and} \quad \mathbf{bi\text{-}IPC} \nvdash p \vee \neg p.$$

- **bi-IPC** $\subseteq$ **L** $\subseteq$ **CPC** is a bi-intermediate logic when it is closed under:
    1. modus ponens,
    2. uniform substitutions,
    3. double negations $\dfrac{\varphi}{\neg\sim\varphi}$.



The lattice $\Lambda(\mathbf{bi\text{-}IPC})$ of bi-intermediate logics

The expressivity of **bi-IPC**

- Thanks to the co-implication, **bi-IPC** gains a lot of expressive power.

Example

Gödel's translation of **IPC** into **S4** extends to a translation of **bi-IPC** into tense-**S4**.

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Example

IPC is Kripke complete w.r.t. the class of posets and to that of trees.

bi-IPC is Kripke complete w.r.t. the class of posets but not to that of trees.

Bi-Heyting algebras

Bi-Heyting algebra

A Heyting algebra **A** equipped with an additional binary operation $\leftarrow$ which satisfies the dual residuation law: for all $a, b, c \in A$,

$$a \leftarrow b \leq c \iff a \leq b \vee c.$$

In other words, **A** is both a Heyting algebra and a co-Heyting algebra.

Theorem

The class **bi-HA** *of bi-Heyting algebras forms a variety that algebraizes* **bi-IPC***.*

Bi-Esakia spaces

Bi-Esakia space

An ordered topological space $\mathcal{X} = (X, \tau, \leq)$ satisfying:

- $\mathcal{X}$ is compact;
- PSA: $\forall x, y \in X \; \big(x \not\leq y \implies \exists\, U \in ClopUp(\mathcal{X}) \; (x \in U \text{ and } y \notin U)\big)$;
- $\forall U \in Clop(\mathcal{X}) \; \big(\downarrow U \in Clop(\mathcal{X})\big)$;
- $\forall U \in Clop(\mathcal{X}) \; \big(\uparrow U \in Clop(\mathcal{X})\big)$.

Bi-Esakia morphisms

Bi-p-morphism

A map $f\colon (X,\leq) \to (Y,\leq)$ between posets satisfying:

- Order preserving: $\forall x, z \in X \ \big(x \leq z \implies f(x) \leq f(z)\big)$;
- Up: $\forall x \in X, \forall y \in Y \ \big(f(x) \leq y \implies \exists\, z \geq x \ (f(z) = y)\big)$;
- Down: $\forall x \in X, \forall y \in Y \ \big(y \leq f(x) \implies \exists\, z \leq x \ (f(z) = y)\big)$.

Bi-Esakia morphism

A continuous bi-p-morphism $f\colon \mathcal{X} \to \mathcal{Y}$ between bi-Esakia spaces.

Bi-Esakia duality

Theorem (L. Esakia 1975)

The category of bi-Heyting algebras and bi-Heyting homomorphisms is dually equivalent to that of bi-Esakia spaces and bi-Esakia morphisms.

- If $\mathbf{A} \in \mathbf{bi\text{-}HA}$, then $\mathbf{A}_* := (\mathit{Spec}(\mathbf{A}), \tau, \subseteq) \in \mathbf{bi\text{-}ESA}$ is the bi-Esakia dual of $\mathbf{A}$.

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- If $\mathcal{X} \in \textbf{bi-ESA}$, then $\mathcal{X}^* := (ClopUp(\mathcal{X}), \cup, \cap, \to, \leftarrow, \emptyset, X) \in \textbf{bi-HA}$ is the algebraic dual of $\mathcal{X}$, where

$$U \to V := X \setminus {\downarrow}(U \setminus V) \qquad \text{and} \qquad U \leftarrow V := {\uparrow}(U \setminus V).$$

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- For $\varphi \in Form$,

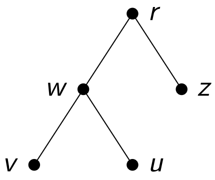
$$\mathcal{X} \models \varphi \iff \mathcal{X}^* \models \varphi$$

The bi-intuitionistic Gödel-Dummett logic

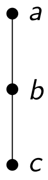
Co-trees and co-forests

Co-tree

A poset $\mathcal{X} = (X, \leq)$ with a greatest element, called the co-root, and whose principal upsets (i.e., sets of the form $\uparrow x$) are chains.



The co-tree $\mathfrak{F}_1$

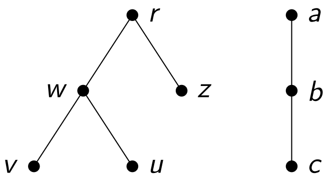


The co-tree $\mathfrak{F}_2$

Co-trees and co-forests

Co-forest

A (possibly empty) disjoint union of co-trees.



The co-forest $\mathfrak{F} = \mathfrak{F}_1 \uplus \mathfrak{F}_2$

Intuitionistic linear calculus

- The formula $(p \to q) \vee (q \to p)$ is called the Gödel-Dummett prelinearity axiom.
- Over **IPC**, it axiomatizes the intuitionistic linear calculus

$$\mathbf{LC} := \mathbf{IPC} + (p \to q) \vee (q \to p).$$

- **LC** is algebraized by the variety of Gödel algebras

$$\mathbf{GA} := \{\mathbf{A} \in \mathbf{HA} \colon \mathbf{A} \models (p \to q) \vee (q \to p)\}.$$

- **LC** is Kripke complete w.r.t. the class of co-trees and to that of chains.
- The lattice $\Lambda(\mathbf{LC})$ of consistent extensions of **LC** is a chain of order type $(\omega + 1)^\partial$ and all of its elements are locally tabular.

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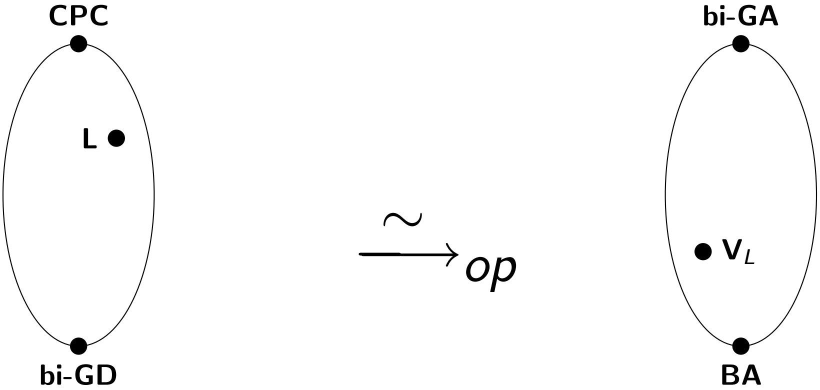
- **bi-GD** is Kripke complete w.r.t. the class of co-trees ~~and to that of chains~~.
- The lattice $\Lambda(\mathbf{bi\text{-}GD})$ of consistent extensions of **bi-GD** is not a chain, has the size of the continuum, and contains only denumerably many locally tabular elements.

The lattices $\Lambda(\textbf{bi-GD})$ and $\Lambda(\textbf{bi-GA})$



The lattice $\Lambda(\textbf{bi-GD})$ of consistent extensions of **bi-GD**

The lattice $\Lambda(\textbf{bi-GA})$ of nontrivial subvarieties of **bi-GA**

Bi-Gödel algebras and co-trees

Proposition

- $\mathbf{A}$ is a bi-Gödel algebra $\iff$ $\mathbf{A}_*$ is a bi-Esakia co-forest;
- $\mathbf{A}$ is a SI bi-Gödel algebra $\iff$ $\mathbf{A}_*$ is a bi-Esakia co-tree.

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Subframe Lemma

Let $\mathcal{Y}$ be bi-Esakia co-forest. If $\mathcal{X}$ is a finite co-tree, then its subframe formula $\beta(\mathcal{X})$ satisfies:

$$\mathcal{Y} \models \beta(\mathcal{X}) \iff \nexists \text{ order embedding } f\colon (X, \leq) \hookrightarrow (Y, \leq)$$
$$\iff \mathcal{X} \not\rightarrowtail \mathcal{Y}.$$

Co-trees, trees, and chains

- The bi-intermediate logic of co-trees is

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- The bi-intermediate logic of trees is

$$\mathbf{bi\text{-}GD}^{\partial} := \mathbf{bi\text{-}IPC} + \neg[(q \leftarrow p) \wedge (p \leftarrow q)].$$

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- The bi-intermediate logic of chains, or the bi-intuitionistic linear calculus, is

$$\mathbf{bi\text{-}LC} := \mathbf{bi\text{-}IPC} + [(p \to q) \vee (q \to p)] + \neg[(q \leftarrow p) \wedge (p \leftarrow q)]$$

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$$\mathbf{bi\text{-}LC} := \mathbf{bi\text{-}IPC} + [(p \to q) \vee (q \to p)] + \neg[(q \leftarrow p) \wedge (p \leftarrow q)]$$

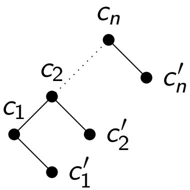
$= \mathbf{bi\text{-}GD} + \beta($  $)$.

Locally finite varieties of bi-Gödel algebras

The finite combs

Define:

- for each $n \in \mathbb{Z}^+$, the n-comb $\mathfrak{C}_n$ as the finite bi-Esakia co-tree



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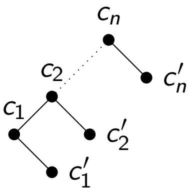

- the variety (generated by the algebraic duals) of the finite combs by

$$\mathbf{V}_{FC} := \mathbb{V}\big(\{\mathfrak{C}_n^* \colon n \in \mathbb{Z}^+\}\big).$$

Locally finite varieties of bi-Gödel algebras

Theorem (N. Bezhanishvili & M. M. & T. Moraschini 2024)

The lattice $\Lambda(\mathbf{bi\text{-}GA})$ *has the size of the continuum and if* $\mathbf{V} \in \Lambda(\mathbf{bi\text{-}GA})$, *then*

$$\mathbf{V} \textit{ is locally finite } \iff \mathbf{V}_{FC} \nsubseteq \mathbf{V} \iff \mathbf{V} \models \beta(\mathfrak{C}_n) \textit{ for some } n \in \mathbb{Z}^+.$$

Consequently, $\mathbf{V}_{FC}$ *is the only pre-locally finite variety of bi-Gödel algebras.*

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Theorem (M. M. & T. Moraschini 2025)

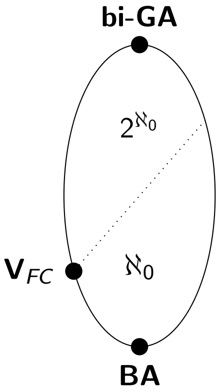
The problem of determining whether a finitely axiomatizable variety of bi-Gödel algebras is locally finite is decidable.

Counting the locally finite varieties of bi-Gödel algebras

The lattice $\Lambda(\mathbf{bi\text{-}GA})$

Theorem

While $\Lambda(\mathbf{bi\text{-}GA})$ has the size of the continuum, it contains only denumerably many locally finite varieties, all of which are finitely axiomatizable.



Counting the locally finite varieties

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$$\left|\{\mathbf{V} \in \Lambda(\mathbf{bi\text{-}GA}) : \mathbf{V} \text{ is locally finite}\}\right| = \Big| \bigcup_{n \in \mathbb{Z}^+} \{\mathbf{V} \in \Lambda(\mathbf{bi\text{-}GA}) : \mathbf{V} \models \beta(\mathfrak{C}_n)\}\Big|$$

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- $\mathbf{V}_{\beta(\mathfrak{C}_n)} \coloneqq \{\mathbf{A} \in \mathbf{bi\text{-}GA} \colon \mathbf{A} \models \beta(\mathfrak{C}_n)\}$.

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Counting the locally finite varieties

Desired Lemma

If $n \in \mathbb{Z}^+$ and $\mathbf{V}_{\beta(\mathfrak{C}_n)} := \{\mathbf{A} \in \textbf{bi-GA} \colon \mathbf{A} \models \beta(\mathfrak{C}_n)\}$, then $|\Lambda(\mathbf{V}_{\beta(\mathfrak{C}_n)})| \leq \aleph_0$.

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Due to many things, this result is equivalent to:

Desired Lemma V2

If $n \in \mathbb{Z}^+$, then

$$\mathcal{T}_n := \{\mathcal{X} \in \mathsf{coTree}^{<\omega} \colon \mathfrak{C}_n \not\twoheadrightarrow \mathcal{X}\}/\cong$$

has no infinite antichains w.r.t. the bi-p-morphic image relation $\leq_p$.

$$\begin{aligned} \mathcal{X} \leq_p \mathcal{Y} &\iff \exists \text{ surjective bi-p-morphism } f \colon \mathcal{Y} \twoheadrightarrow \mathcal{X} \\ &\iff \mathcal{Y} \twoheadrightarrow \mathcal{X}. \end{aligned}$$

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- <u>Case $n = 1$:</u> Trivial because $\mathcal{T}_1 = \{\bullet\}$.

$$\mathfrak{C}_1 \not\hookrightarrow \mathcal{X}$$

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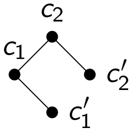
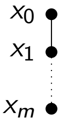
- <u>Case $n = 2$:</u>

$$\mathfrak{C}_2 \not\hookrightarrow \mathcal{X}$$





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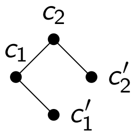

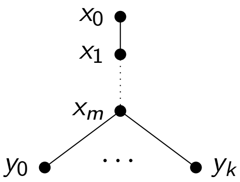

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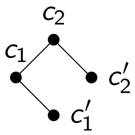

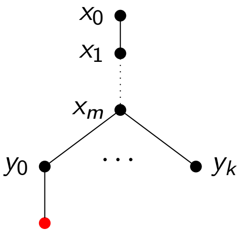

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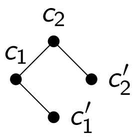
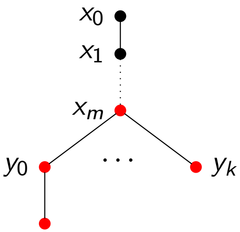
If $n \in \mathbb{Z}^+$, then $\mathcal{T}_n = \{\mathcal{X} \in \text{coTree}^{<\omega} : \mathfrak{C}_n \not\hookrightarrow \mathcal{X}\}/\cong$ has no infinite $\leq_p$-antichains.

- <u>Case $n = 2$:</u>

$$\mathfrak{C}_2 \hookrightarrow \mathcal{X}$$





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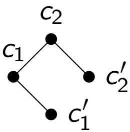

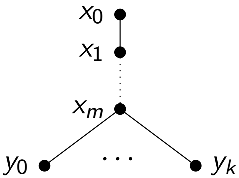

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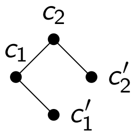

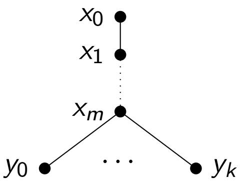

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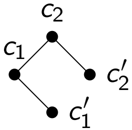
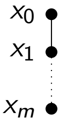
$$\mathfrak{C}_2 \not\hookrightarrow \mathcal{X} = \tau(m, 0)$$





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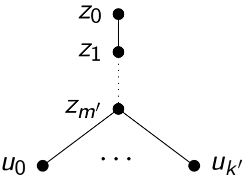

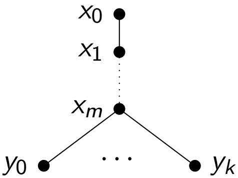

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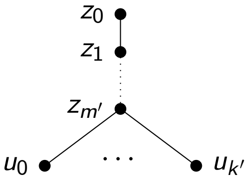

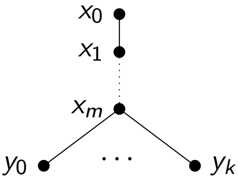

$$\mathcal{Z} \twoheadrightarrow \mathcal{X} \iff \tau(m, k) \leq_p \tau(m', k') \iff m \leq_\omega m' \;\&\; k \leq_\omega k' \iff (m, k) \leq_{\omega^2} (m', k').$$

Counting the locally finite varieties

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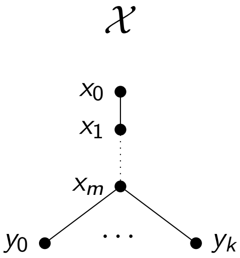
- Case $n = 2$:

The map $\tau\colon (\omega, \leq) \times (\omega, \leq) \to (\mathcal{T}_2, \leq_p)$ defined by

$$\tau(m, k) := \begin{cases} \text{the chain } \{x_0, \ldots, x_m\} & \text{if } k = 0, \\ \text{the co-tree } \mathcal{X} & \text{if } k > 0, \end{cases}$$

is an order isomorphism.



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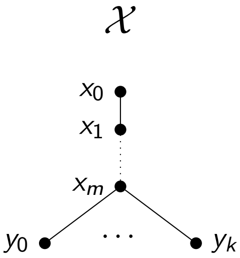

Since $(\omega, \leq)$ is a well order, $(\mathcal{T}_2, \leq_p)$ is a well partial order.

In particular, $\mathcal{T}_2$ has no infinite $\leq_p$-antichains.

Counting the locally finite varieties

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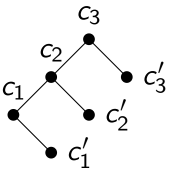

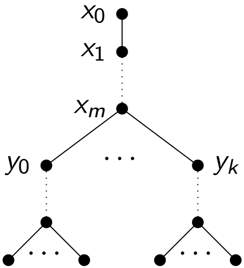

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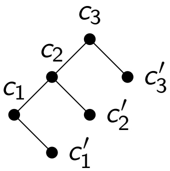

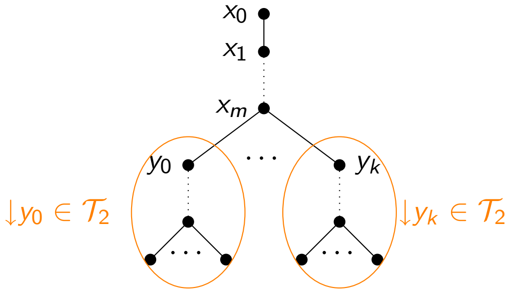

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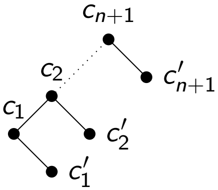
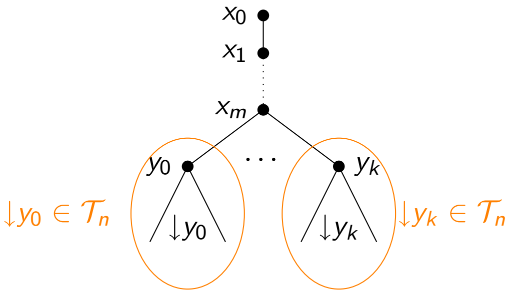
- Case $n + 1$:

$$\mathfrak{C}_{n+1} \not\hookrightarrow \mathcal{X}$$





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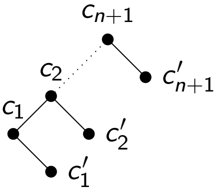

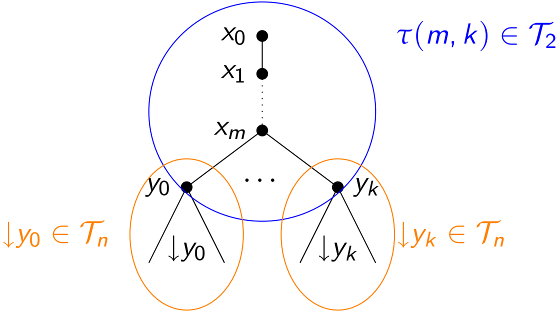

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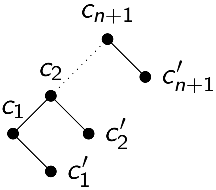
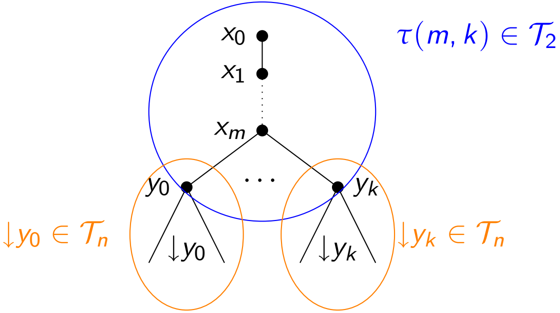
- <u>Case $n+1$:</u>

$$\mathfrak{C}_{n+1} \not\hookrightarrow \mathcal{X} \equiv (\tau(m,k), [\downarrow y_0, \ldots, \downarrow y_k]) \in \mathcal{T}_2 \times \mathcal{T}_n^{\#}$$





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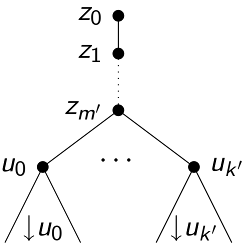

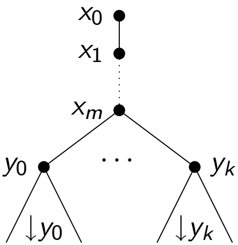

Counting the locally finite varieties

$$(\tau(m', k'), [\downarrow u_0, \ldots, \downarrow u_{k'}]) \equiv \mathcal{Z} \twoheadrightarrow \mathcal{X} \equiv (\tau(m, k), [\downarrow y_0, \ldots, \downarrow y_k]) \in \mathcal{T}_2 \times \mathcal{T}_n^{\#}$$

We have $\mathcal{Z} \twoheadrightarrow \mathcal{X}$ when $\tau(m, k) \leq_p \tau(m', k')$ and there exists a surjective map

$$h\colon [\downarrow u_0, \ldots, \downarrow u_{k'}] \twoheadrightarrow [\downarrow y_0, \ldots, \downarrow y_k] \quad \text{s.t.} \quad h(\downarrow u_i) \leq_p \downarrow u_i \ \text{ for every } i \leq k'.$$

Denote the latter condition by $[\downarrow y_0, \ldots, \downarrow y_k] << [\downarrow u_0, \ldots, \downarrow u_{k'}]$.

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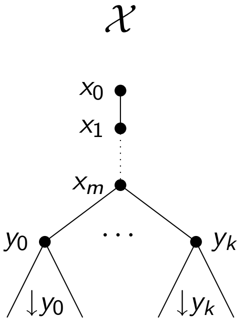
- Case $n+1$:

The map

$$\begin{aligned}\pi \colon (\mathcal{T}_{n+1}, \leq_p) &\to (\mathcal{T}_2, \leq_p) \times (\mathcal{T}_n^{\#}, <<) \\ \mathcal{X} &\mapsto \big(\tau(m,k), [{\downarrow}y_0, \ldots, {\downarrow}y_k]\big)\end{aligned}$$

is order reflecting.



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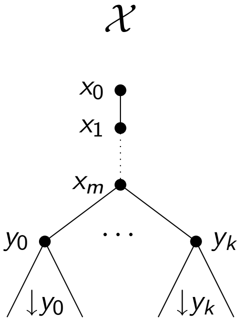

Easy Fact

If a map $f \colon P \to Q$ between posets is order reflecting and Q has no infinite antichains, then neither does P.

Counting the locally finite varieties

Multiset Projectivity Relation $<<$

Given a poset $(P, \leq)$, we define $<<$ on the set $P^{\#}$ of finite multisets of P by:

$$[p_1, \dots, p_k] << [q_1, \dots, q_{k'}] \iff \exists\, h\colon [q_1, \dots, q_{k'}] \twoheadrightarrow [p_1, \dots, p_k] \text{ s.t. } \forall i \leq k' \;\; \big(h(q_i) \leq q_i\big).$$

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Remark

$<<$ is strictly stronger than the Higman's multiset embeddability relation:

$$[p_1, \ldots, p_k] \preceq [q_1, \ldots, q_{k'}] \iff \exists\, g\colon [p_1, \ldots, p_k] \hookrightarrow [q_1, \ldots, q_{k'}] \text{ s.t. } \forall i \leq k \ \big(p_i \leq g(p_i)\big).$$

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Theorem

If $(P, \leq)$ is a better partial order (BPO for short), then so is $(P^{\#}, <<)$.

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If $n \in \mathbb{Z}^+$, then $\mathcal{T}_n = \{\mathcal{X} \in \mathrm{coTree}^{<\omega} \colon \mathfrak{C}_n \not\to \mathcal{X}\}/\cong$ has no infinite $\leq_p$-antichains.

- <u>Case $n+1$:</u> Assume that $(\mathcal{T}_n, \leq_p)$ is a BPO, hence so is $(\mathcal{T}_n^{\#}, <<)$.

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Since $(\omega, \leq) \times (\omega, \leq) \cong (\mathcal{T}_2, \leq_p)$, Facts 1 & 2 ensure that $(\mathcal{T}_2, \leq_p) \times (\mathcal{T}_n^{\#}, <<)$ is a BPO.

Fact 1

Well orders are BPOs.

Fact 2

Products of BPOs are BPOs.

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Since the map

$$\begin{aligned} \pi \colon (\mathcal{T}_{n+1}, \leq_p) &\to (\mathcal{T}_2, \leq_p) \times (\mathcal{T}_n^{\#}, <<) \\ \mathcal{X} &\mapsto \big(\tau(m,k), [\downarrow y_0, \dots, \downarrow y_k]\big) \end{aligned}$$

is order reflecting, Fact 3 entails that $(\mathcal{T}_{n+1}, \leq_p)$ must be a BPO.

Fact 3

If a map $f \colon P \to Q$ between posets is order reflecting and Q is a BPO, then P is also a BPO.

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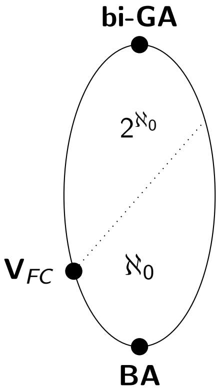
is order reflecting, Fact 3 entails that $(\mathcal{T}_{n+1}, \leq_p)$ must be a BPO.

In particular, $(\mathcal{T}_{n+1}, \leq_p)$ lacks infinite antichains. $\square$

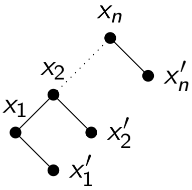
The lattice $\Lambda(\textbf{bi-GA})$

Theorem

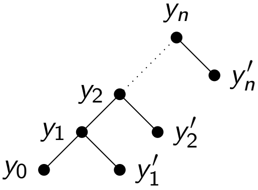
While $\Lambda(\textbf{bi-GA})$ has the size of the continuum, it contains only denumerably many locally finite varieties, all of which are finitely axiomatizable.



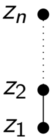
The lattice $\Lambda(\mathbf{bi\text{-}GA})$



the n-comb $\mathfrak{C}_n$

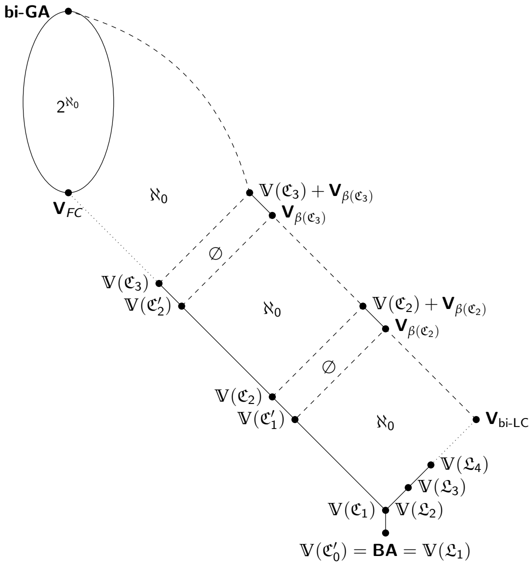


the n-hcomb $\mathfrak{C}_n'$



the n-chain $\mathfrak{L}_n$

The lattice $\Lambda(\mathbf{bi\text{-}GA})$



Thank you for your attention!