# On the Decidability of Comparing Many-Valued Logics: A WSkS Approach

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## From matrices to partial non-deterministic matrices

- Non-determinism (1980s Kearns, Lev; 2001, Avron *et al.*)
- Partiality (2013, Baaz, Lahav, Zamansky)
- Finite-valued semantics for logics not characterizable by finite matrices
- Compositionality results and bridge with analytic calculi

**Definition**

A PNmatrix over a signature $\Sigma$ is a tuple $\mathbb{M} = \langle A, \cdot_{\mathbb{M}}, D \rangle$

- $A$ is the set of truth-values,
- $D \subseteq A$ is the set of designated truth-values,
- for every $© \in \Sigma^{(n)}$, $©_{\mathbb{M}} : A^n \to \wp(A)$ is the interpretation of $©$ in $\mathbb{M}$.

*Total* if the $©_{\mathbb{M}}(\vec{a})$ are non-empty (Nmatrix), *partial* (Pmatrix) if either singletons or empty and *deterministic* (or matrix) if all singletons.

A valuation $v \in \mathrm{Val}(\mathbb{M})$ is a function $v : \mathrm{Fm} \to A$ s.t.

$$v(©(\varphi_1, \ldots, \varphi_n)) \in ©_{\mathbb{M}}(v(\varphi_1), \ldots, v(\varphi_n)).$$

Partiality restricts which sets of values can coexist in one valuation.

# PNmatrix Semantics

**Set $\times$ Set: Scottian consequence relation**

$$\Gamma \rhd_{\mathbb{M}} \Delta$$

iff no valuation $v$ over $\mathbb{M} = \langle A, \cdot_{\mathbb{M}}, D\rangle$ designates every premise in $\Gamma$ while designating no conclusion in $\Delta$:

$$v(\Gamma) \subseteq D \quad \text{and} \quad v(\Delta) \subseteq A \setminus D.$$

**Thm and Set $\times$ Fmla as fragments**

- Theorems (Thm): $\mathrm{Thm}_{\mathbb{M}} = \{\varphi : \emptyset \rhd_{\mathbb{M}} \{\varphi\}\}$
- Tarskian consequence relation (Set $\times$ Fmla): $\Gamma \vdash_{\mathbb{M}} \varphi \quad \text{iff} \quad \Gamma \rhd_{\mathbb{M}} \{\varphi\}$

# Total components

## Definition

For $X \subseteq A$, let $\mathbb{M}_X$ be the restriction of $\mathbb{M}$ to $X$:

$$\copyright_{\mathbb{M}_X}(a_1, \dots, a_k) := \copyright_{\mathbb{M}}(a_1, \dots, a_k) \cap X.$$

A non-empty $X$ is a *total component* if every entry of $\mathbb{M}_X$ is non-empty.

## PNmatrices pack Nmatrices

A PNmatrix packs a family of Nmatrices together: an empty entry simply forbids a valuation from crossing between components.

$$\rhd_{\mathbb{M}} = \bigcap_{X \in \mathsf{TotComp}(\mathbb{M})} \rhd_{\mathbb{M}_X}, \qquad \vdash_{\mathbb{M}} = \bigcap_{X} \vdash_{\mathbb{M}_X}, \qquad \mathsf{Thm}_{\mathbb{M}} = \bigcap_{X} \mathsf{Thm}_{\mathbb{M}_X}.$$

## Subcase: deterministic total components

If every $\mathbb{M}_X$, with $X \in \mathsf{TotComp}(\mathbb{M})$, is deterministic, then $\mathbb{M}$ is just packaging a finite family of finite matrices.

# Two finite-valued PNmatrices

## Modal logic without possible worlds (Kearns 1981)

$$\mathbb{K} = \langle \{F, f, t, T\}, \cdot_{\mathbb{K}}, \{t, T\} \rangle$$

| $\vee_{\mathbb{K}}$ | $F$ | $f$ | $t$ | $T$ |
|---|---|---|---|---|
| $F$ | $F$ | $f$ | $t$ | $T$ |
| $f$ | $f$ | $f$ | $t, T$ | $T$ |
| $t$ | $t$ | $t, T$ | $t, T$ | $T$ |
| $T$ | $T$ | $T$ | $T$ | $T$ |

| $\to_{\mathbb{K}}$ | $F$ | $f$ | $t$ | $T$ |
|---|---|---|---|---|
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $f$ | $t$ | $t, T$ | $t, T$ | $T$ |
| $t$ | $f$ | $f$ | $t, T$ | $T$ |
| $T$ | $F$ | $f$ | $t$ | $T$ |

| | $\neg_{\mathbb{K}}$ | $\Box_{\mathbb{K}}$ |
|---|---|---|
| $F$ | $T$ | $F, f$ |
| $f$ | $t$ | $F, f$ |
| $t$ | $f$ | $F, f$ |
| $T$ | $F$ | $t, T$ |

## Processors dealing with multiple-sources (Avron&Ben-Naim&Konikowska 2007)

$$\mathbb{S} = \langle \{f, \bot, \top, t\}, \cdot_{\mathbb{S}}, \{\top, t\} \rangle$$

| $\wedge_{\mathbb{S}}$ | $f$ | $\bot$ | $\top$ | $t$ |
|---|---|---|---|---|
| $f$ | $f$ | $f$ | $f$ | $f$ |
| $\bot$ | $f$ | $f, \bot$ | $f$ | $f, \bot$ |
| $\top$ | $f$ | $f$ | $\top$ | $\top$ |
| $t$ | $f$ | $f, \bot$ | $\top$ | $t, \top$ |

| $\vee_{\mathbb{S}}$ | $f$ | $\bot$ | $\top$ | $t$ |
|---|---|---|---|---|
| $f$ | $f, \top$ | $t, \bot$ | $\top$ | $t$ |
| $\bot$ | $t, \bot$ | $t, \bot$ | $t$ | $t$ |
| $\top$ | $\top$ | $t$ | $\top$ | $t$ |
| $t$ | $t$ | $t$ | $t$ | $t$ |

| | $\neg_{\mathbb{S}}$ |
|---|---|
| $f$ | $t$ |
| $\bot$ | $\bot$ |
| $\top$ | $\top$ |
| $t$ | $f$ |

## $\rhd_{\mathbb{S}} \neq \rhd_{\mathbb{K}}$

### Comparing their $\{\vee\}$-fragment

Take $\Upsilon = \{p, q, p \vee q\}$, finite and subformula-closed.

Green $=$ designated, Red $=$ undesignated.

Try the valuation $v(p) = v(q) = f$:

$$\begin{tikzcd} & p \vee q : \top \arrow[dl] \arrow[dr] & \\ p : f & & q : f \end{tikzcd}$$

- $p \vee q \not\rhd_{\mathbb{S}} p, q$: $D_{\mathbb{S}} = \{\top, t\}$ and $f \vee_{\mathbb{S}} f = \{f, \top\}$, so take $v(p \vee q) = \top$.
- $p \vee q \rhd_{\mathbb{K}} p, q$: $D_{\mathbb{K}} = \{t, T\}$; with $v(p), v(q) \in \{F, f\}$,

| $\vee_{\mathbb{K}}$ | $F$ | $f$ |
|---|---|---|
| $F$ | $F$ | $f$ |
| $f$ | $f$ | $f$ |

keeps $v(p \vee q) \in \{F, f\} \notin D_{\mathbb{K}}$.

# Motivation: when does $\rhd_{\mathbb{M}} \subseteq \rhd_{\mathbb{M}/\equiv}$? (Set $\times$ Set)

## $\mathcal{M}$ a class of matrices [Czelakowski 1983]

$$\mathsf{Matr}(\rhd_{\mathcal{M}}) = \overleftarrow{H_s} H_s S P_u(\mathcal{M}).$$

$H_s$: strict homomorphic images (quotients by a congruence compatible with designation). $\overleftarrow{H_s}$: strict homomorphic preimages. $S$: submatrices. $P_u$: ultraproducts.

- $H_s$-image is a quotient by an (operationally well-understood) congruence $\equiv$, and $\rhd_{\mathbb{M}} = \rhd_{\mathbb{M}/\equiv}$

## $\mathcal{M}$ a class of Nmatrices [Caleiro, M & Rivieccio 2023]

$$\mathsf{Nmatr}(\rhd_{\mathcal{M}}) = H_s^{\rhd} \, \overleftarrow{H_s} \, S \, P_u(\mathcal{M}).$$

- In Nmatrices it makes sense to make quotients by arbitrary equivalence relations $\equiv$ compatible with designated values.
- Always have $\rhd_{\mathbb{M}/\equiv} \subseteq \rhd_{\mathbb{M}}$ but not necessarily $\rhd_{\mathbb{M}/\equiv} = \rhd_{\mathbb{M}}$
- $H_s$ is replaced by $H_s^{\rhd} := \{\rhd\text{-sound images}\}$.

# Congruences, where art thou?

## For matrices

On a matrix the congruences compatible with the designated values form a lattice with a maximum (the Leibniz congruence), giving a unique reduced model.

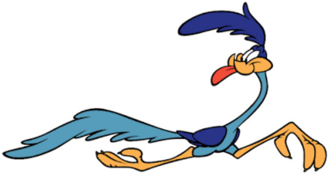
## Nmatrices: this breaks

$\exists\,\mathbb{M}$ with $\mathbb{M}/{\equiv_{01}}$, $\mathbb{M}/{\equiv_{12}}$ both $\rhd_{\mathbb{M}}$-sound, but no $\equiv\,\supseteq\,\equiv_{01}\cup\equiv_{12}$ stays compatible *and* sound.

- No join, no canonical max: "the" reduced model doesn't exist.
- Still, we want the **generalization of the Leibniz operator**: for matrices it gives one reduced model, but for (P)Nmatrices the set of maximal sound $\equiv$'s (Nmatrix congruences) and their quotients $\mathbb{M}/{\equiv}$ (the reduced models). The comparison problem is a first step.

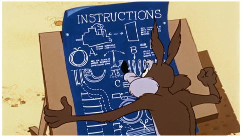
# The problem: comparing logics defined by finite PNmatrices

Given two finite semantic presentations $\mathbb{M}_1, \mathbb{M}_2$, can we effectively decide whether the logic defined by $\mathbb{M}_1$ is included in the logic defined by $\mathbb{M}_2$?



1. Parameters: the notion of logic $(\mathrm{Thm}, \mathrm{Set} \times \mathrm{Fmla}, \mathrm{Set} \times \mathrm{Set})$ and of the structures used to present it ((P)(N)matrices).
2. For Finite logical matrices: comparison decidable.
3. With non-determinism: Thm-, Set $\times$ Fmla-comparison undecidable; Set $\times$ Set-comparison OPEN.

# Spoiler: Good problems bite back




Plan: encode $\triangleright_{\mathbb{M}_1} \overset{?}{\subseteq} \triangleright_{\mathbb{M}_2}$ in WSkS.

Implement the encoding in a WSkS solver (MONA).

Wait: SameForm is not WSkS.

Decidability direction: WSkS reduction was wrong, we found out when **implementing** in MONA.


Undecidability direction: We have not been able to adapt the strategies for undecidability of $\mathsf{Thm}, \mathsf{Set} \times \mathsf{Fmla}$ to $\mathsf{Set} \times \mathsf{Set}$.

# Finite logical matrices: folklore

## Finite set of formulas suffices

For finite logical matrices $\mathbb{M}_1 = \langle A_1, \cdot_{\mathbb{M}_1}, D_1\rangle$ and $\mathbb{M}_2 = \langle A_2, \cdot_{\mathbb{M}_2}, D_2\rangle$, let

$$m = \max(|A_1|, |A_2|).$$

To compare their induced logics, it is enough to test formulas in $\mathsf{Fm}_m$.
For $\varphi \in \mathsf{Fm}_m$, let $[\varphi]_i$ be its term-function in $\mathbb{M}_i$. Finitely many pairs $([\varphi]_1, [\varphi]_2)$ exist, so closing $P_m$ under the connectives, adding a formula only for new pairs, terminates in a finite $\Theta \subseteq \mathsf{Fm}_m$ with

$$\forall\varphi \in \mathsf{Fm}_m\ \exists\varphi^* \in \Theta : \ \varphi \dashv\vdash_1 \varphi^* \text{ and } \varphi \dashv\vdash_2 \varphi^*.$$

For comparing Thm ($\emptyset \rhd_i \varphi$), Set $\times$ Fmla ($\Gamma \rhd_i \varphi$), and Set $\times$ Set ($\Gamma \rhd_i \Delta$) testing $\varphi, \Gamma, \Delta \subseteq \Theta$ is enough.

## Set $\times$ Set-logics seem easier to compare (Czelakowski 1983)

For finite $\mathbb{M}_1, \mathbb{M}_2$,

$$\rhd^{\mathsf{Set}\times\mathsf{Set}}_{\mathbb{M}_1} = \rhd^{\mathsf{Set}\times\mathsf{Set}}_{\mathbb{M}_2} \quad\Longleftrightarrow\quad \mathbb{M}_1^* \cong \mathbb{M}_2^*.$$

## Non-determinism breaks the argument

**Deterministic case: local tabularity**

For a finite matrix $\mathbb{M}$, $\mathsf{Fm}_n/\dashv\vdash_{\mathbb{M}}$ is finite.
$\varphi$ induces a term-*function* $[\varphi]_i$. If $[\varphi]_i = [\psi]_i$ (same function), then $\varphi \dashv\vdash_i \psi$.

**Non-determinism: local tabularity lost...**

For $\varphi \in \mathsf{Fm}_n$ with variables $p_1, \ldots, p_n$, let its *multifunction* $[\varphi]_{\mathbb{M}} : A^n \to \wp(A)$ be

$$[\varphi]_{\mathbb{M}}(a_1, \ldots, a_n) = \{v(\varphi) : v \in \mathsf{Val}(\mathbb{M}),\ v(p_i) = a_i,\ 1 \leq i \leq n\}.$$

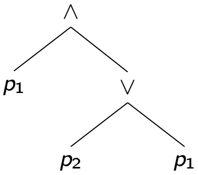
Then $[\varphi]_{\mathbb{M}} = [\psi]_{\mathbb{M}}$ *need not* give $\varphi \dashv\vdash_{\mathbb{M}} \psi$: $\mathsf{Fm}_n/\dashv\vdash_{\mathbb{M}}$ not always finite. Worse

$$[\varphi \odot \psi]_{\mathbb{M}}(\vec{a}) \ \subseteq \bigcup_{x \in [\varphi]_{\mathbb{M}}(\vec{a}),\ y \in [\psi]_{\mathbb{M}}(\vec{a})} \odot_{\mathbb{M}}(x, y),$$

possibly *strictly*; depends on shared subformulas, syntax dependent.

# Formulas as Trees

- If maximum arity of the connectives is $k$, then the set of all formulas can be seen as a $k$-ary tree (or forest).
- More generally, a finite *set of formulas closed under subformulas* is encoded by marking finitely many such tree positions; repeated subformulas may occur at different nodes.




We **decorate** the tree using various predicates:

- one for each variable $\mathsf{P}_i$ for $i \in \{1, \ldots, n\}$
- one for each connective, $\mathsf{C}_f$ for $f \in \Sigma$
- one for each truth value (under a valuation), $\mathsf{V}_a$ for $a \in A$
- one for the associated bivaluation, $\mathsf{B}$

## The WSkS strategy

Recall: formulas are labelled trees ($\mathsf{P}_i, \mathsf{C}_f$); valuations and bivaluations add labels ($\mathsf{V}_a, \mathsf{B}$)

- For a bivaluation $b$, write $\mathcal{B}_\Upsilon = \{b_{|\Upsilon} : b \in \mathcal{B}\}$. Then

$$\rhd^{|n}_{\mathbb{M}_1} \subseteq \rhd^{|n}_{\mathbb{M}_2} \quad \text{iff} \quad (\mathcal{B}_{\mathbb{M}_2})_\Upsilon \subseteq (\mathcal{B}_{\mathbb{M}_1})_\Upsilon \quad \text{for every finite } \Upsilon = \mathrm{sub}(\Upsilon) \subseteq \mathsf{Fm}_n.$$

- Shared vocabulary is only $\mathsf{P}_i, \mathsf{C}_f, \mathsf{B}$ ($\mathsf{V}_a$ is internal to each $\mathbb{M}_i$, witnessed existentially).
- **The question:** are the $\mathbb{M}_2$-labelled trees contained in the $\mathbb{M}_1$-labelled trees?

**Why WSkS**

Logic of labelled $k$-ary trees, quantifying over **finite sets of nodes** (MSO); **decidable** (Rabin, 1969). The predicate vocabulary is finite , $\mathsf{P}_i$ ($n$ variables), $\mathsf{C}_f$ ($f \in \Sigma$), $\mathsf{V}_a$ ($a \in A$), $\mathsf{B}$. Only the number of variables $n$ is unbounded, so it must be *fixed* before the encoding is written.

## Everything we need to encode

A labelling of the tree must be checkable, in WSkS, as a legal $\Upsilon$-bivaluation pattern:

- **Well-formed formulas**: each node has exactly one $\mathsf{P}_i$ or $\mathsf{C}_f$, arities match.
- **Legal valuation**: at a $\mathsf{C}_f$-node with children $x_1, \dots, x_k$, $\mathsf{V}_a$ holds only for $a \in f_{\mathbb{M}}(a_1, \dots, a_k)$, $a_j$ the child values.
- **Valuation Coherence**: same subformula, same value:

$$\mathsf{SameForm}(x, y) \;\to\; \big(\mathsf{V}_a(x) \leftrightarrow \mathsf{V}_a(y)\big) \text{ for all } a.$$

- **Designation**: $\mathsf{B}(x) \leftrightarrow \bigvee_{a \in D} \mathsf{V}_a(x)$.
- **Inside a total component**: all values forced into a single total component,

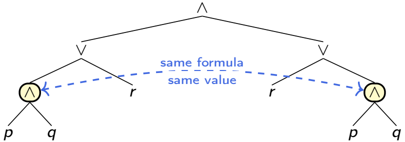
$$\bigvee_{X \in \mathsf{TotComp}(\mathbb{M})} \forall x \bigvee_{a \in X} \mathsf{V}_a(x).$$

**One of these is not like the others**

SameForm was where the encoding broke down.

# Where the encoding breaks

Same subformula $\Rightarrow$ same value; easy for variables $(\bigwedge_{i,a}(\mathsf{P}_i(x) \wedge \mathsf{P}_i(y) \to \mathsf{V}_a(x) \leftrightarrow \mathsf{V}_a(y)))$, but in general:




## The slipped step: SameForm is not a WSkS formula

$$\mathsf{SameForm}(\vec{\mathsf{S}}, x, y) := \bigwedge_{1 \le i \le n} (\mathsf{P}_i(x) \leftrightarrow \mathsf{P}_i(y)) \;\wedge\; \bigwedge_{f \in \Sigma} (\mathsf{C}_f(x) \leftrightarrow \mathsf{C}_f(y)) \;\wedge$$
$$\bigwedge_{1 \le j \le k} ((\mathsf{IsForm}(\vec{\mathsf{S}}, xj) \vee \mathsf{IsForm}(\vec{\mathsf{S}}, yj)) \to \mathsf{SameForm}(\vec{\mathsf{S}}, xj, yj))$$

SameForm appears on *both* sides, recursion, not allowed in WSkS! A genuinely *binary* predicate on nodes seems to be needed, not allowed in WSkS!

**What nondeterminism adds:** with a genuine non-deterministic connective (e.g. $t \wedge_{\mathbb{S}} t = \{t, \top\}$), we need to force two occurrences of the same formula to have the same value, which we failed express within the WSkS.

# Error found during implementation

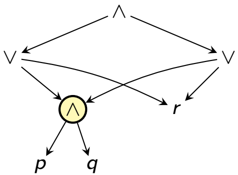
**Not caught on paper**

This was not caught by inspecting the construction, only when implementing in MONA (a concrete WSkS solver).


**When the idea survives**

Exactly when the input PNmatrices are (packaging) a finite set of deterministic matrices, where no binary predicate is needed.

# The undecidability of the Thm and Set $\times$ Fmla comparison




## Dag-Automata universality problem $=$ $\exists$Thm (for Nmatrices)

A term-DAG never repeats an isomorphic closed subgraph. Build $\mathbb{M}_{\mathcal{A}}$ from $\mathcal{A} = \langle \Sigma, Q, F, \delta \rangle$ with $\mathcal{L}(\mathcal{A}) = \mathrm{Fm}(\emptyset) \setminus \mathsf{Thm}(\mathbb{M}_{\mathcal{A}})$, so

$$\mathcal{A} \text{ universal} \iff \mathsf{Thm}(\mathbb{M}_{\mathcal{A}}) = \emptyset = \mathsf{Thm}(\mathbf{2}_{\mathsf{unc}}).$$

Here $\mathbf{2}_{\mathsf{unc}} = \langle \{0,1\}, \cdot_{\mathsf{unc}}, \{1\} \rangle$, $f_{\mathsf{unc}}(\vec{x}) = \{0,1\}$ (no theorems).

## Thm, Set $\times$ Fmla undecidable; Set $\times$ Set a road block

$\exists$Thm is undecidable for Nmatrices, so Thm-comparison is and Set $\times$ Fmla, via a finiteness-preserving operation onto the *smallest* Set $\times$ Fmla-logic with the same theorems. For Set $\times$ Set that operation is *not* minimal: the reduction breaks, Set $\times$ Set stays **open**, perhaps even decidable.

# Corrected positive result

## Fixed-variable Set × Set-comparison

For every $n \in \mathbb{N}$, the problem of deciding whether

$$\rhd^{|n}_{\mathbb{M}_1} \subseteq \rhd^{|n}_{\mathbb{M}_2}$$

is decidable for finite $\Sigma$-PNmatrices $\mathbb{M}_1, \mathbb{M}_2$ whose maximal total components are deterministic.
The WSkS reduction:

$$\Phi^n_{\mathbb{M}_1\mathbb{M}_2} := \forall \vec{S} B. ((\mathsf{IsSetOfForm}(\vec{S}) \land \mathsf{Bival}^{\mathsf{detTC}}_{\mathbb{M}_2}(\vec{S}, B)) \to \mathsf{Bival}^{\mathsf{detTC}}_{\mathbb{M}_1}(\vec{S}, B)).$$

$$\rhd^{|n}_{\mathbb{M}_1} \subseteq \rhd^{|n}_{\mathbb{M}_2} \iff \Phi^n_{\mathbb{M}_1\mathbb{M}_2} \text{ is valid in WSkS.}$$

## What WSkS does here

Checks whether every $\mathbb{M}_2$-bivaluation is also an $\mathbb{M}_1$-bivaluation. $\mathsf{Bival}^{\mathsf{detTC}}$ replaces the formula SameForm by SameVar.

*Bounded-depth variant.* For a fixed depth $k$, one may instead replace SameForm by the finite WSkS unrolling $\mathsf{SameForm}_{\leq k}$, provided IsSetOfForm is also changed so that every formula satisfies $\mathsf{IsForm}_{\leq k}$.

# Packaging Kleene's logic of order into a Pmatrix

## Packaging by partiality

Kleene's logic of order can be presented by a family of deterministic matrices, for example

$$\mathrm{Log}\{\langle K_3, \{1\}\rangle, \langle K_3, \{\tfrac{1}{2}, 1\}\rangle\}.$$

Order-preserving logics record more than preservation of a single designated set.

Package both as one PNmatrix over $\{0, a, b, 1\}$ with $D = \{b, 1\}$: $a$ plays $\frac{1}{2}$ under $D_1$, $b$ plays $\frac{1}{2}$ under $D_2$.

| $\wedge$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $\emptyset$ | $a$ |
| $b$ | $0$ | $\emptyset$ | $b$ | $b$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

| $x$ | $\neg x$ |
|---|---|
| $0$ | $1$ |
| $a$ | $a$ |
| $b$ | $b$ |
| $1$ | $0$ |

$\vee$ mirrors $\wedge$: singleton or $\emptyset$ everywhere, never a real choice.

# Packaging $\mathrm{OL}_3$'s logic of order into a PNmatrix

Every entry of the Kleene PNmatrix above has size $\leq 1$ (only *partiality*, never real choice).
Instead, consider $\mathrm{OL}_3 = \langle\{0, \frac{1}{2}, 1\}; \wedge, \vee, \to, \neg\rangle$: Log$\langle\mathrm{OL}_3, \{\frac{1}{2}\}\rangle$ versus Log$\langle\mathrm{OL}_3, \{\frac{1}{2}, 1\}\rangle$, packaged by doubling 1 into $1^-, 1^+$ with a *single* $D = \{\frac{1}{2}, 1^+\}$.

| $\wedge$ | $0$ | $\frac{1}{2}$ | $1^-$ | $1^+$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1^-$ | $1^+$ |
| $1^-$ | $0$ | $1^-$ | $1^-$ | $\emptyset$ |
| $1^+$ | $0$ | $1^+$ | $\emptyset$ | $1^+$ |

| $x$ | $\neg x$ |
|---|---|
| $0$ | $1^-, 1^+$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1^-$ | $0$ |
| $1^+$ | $0$ |

$\vee, \to$ likewise: singleton or $\emptyset$ everywhere.

**Still "fake" nondeterminism**

$\neg(0) = \{1^-, 1^+\}$ seems genuinely non-deterministic. Yet erasing $1^-$ gives $D \cap X = \{\frac{1}{2}, 1\}$, erasing $1^+$ gives $D \cap X = \{\frac{1}{2}\}$: each a deterministic $\mathrm{OL}_3$ matrix. Both maximal total components are still deterministic.

# Comparing logics of finite sets of finite matrices, still folklore

## Same argument, now simultaneous

Deterministic total components mean $\mathbb{M}_1$ *is*, semantically, a finite family of matrices $\mathbb{M}_{1,1}, \ldots, \mathbb{M}_{1,k_1}$, and likewise $\mathbb{M}_2$ is $\mathbb{M}_{2,1}, \ldots, \mathbb{M}_{2,k_2}$, with $\rhd_{\mathbb{M}_i} = \bigcap_j \rhd_{\mathbb{M}_{i,j}}$. For $\varphi \in \mathsf{Fm}_m$ use the simultaneous tuple

$$[\varphi]^\sharp = ([\varphi]_{1,1}, \ldots, [\varphi]_{1,k_1}, [\varphi]_{2,1}, \ldots, [\varphi]_{2,k_2}),$$

where each coordinate is the term-function of $\varphi$ in that component.
Again, there is a finite $\Theta$ such that every $\varphi \in Fm_m$ has a representative $\varphi^* \in \Theta$ with the same term-function in *every* total component of both PNmatrices.

## Finite variables are enough

In particular,

$$n = \max\{|X| : X \in \mathsf{TotComp}(\mathbb{M}_1) \cup \mathsf{TotComp}(\mathbb{M}_2)\}.$$

Then $\rhd_{\mathbb{M}_1} \subseteq \rhd_{\mathbb{M}_2} \iff \rhd_{\mathbb{M}_1}^{|n} \subseteq \rhd_{\mathbb{M}_2}^{|n}$.

## So what did WSkS actually buy us?

### No new decidability

The reduction only reaches PNmatrices with deterministic maximal total components , i.e. finite families of matrices, where Set $\times$ Set-comparison was *already* decidable by standard means.

### Practical upside

Still, WSkS handles the *packaged* object directly (one MONA run on $\mathbb{M}_1, \mathbb{M}_2$ as given), and the encoding yields benchmark formulas for MONA and other WSkS solvers.

# Final remarks

## Summary

- The WSkS reduction survives exactly when the PNmatrices in play are (packaging) a finite set of deterministic matrices.
- This strategy highlights the connection to automata-theoretic methods and how non-determinism pushes beyond the limits of WSkS.

## So the problem remains open

- **In the positive direction:** For bounded $n$, try to encode the problem in a decidable logic *extending* WSkS. Suggestions?
  - $n = \omega$ would be covered if finite PNmatrices axiomatizable with a computable number of variables, also open! (true for monadic ones: $\mathsf{maxArity}(\Sigma) + 1$).
  - Finding the right notion of *congruence* for PNmatrices would need just the subproblem $\rhd_{\mathbb{M}} \overset{?}{\subseteq} \rhd_{\mathbb{M}/\equiv}$ decidable. Can it be easier?
- **In the negative direction:** Try to extend the DAG-automata/Counter Machines undecidability strategy to Set $\times$ Set...pick other undecidable problems? Suggestions?

Thank you!