Quasiequational Bases for Oriented Paths

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Quasivarieties of relational structures

Definition

Let τ be a relational signature.

- A quasiequation is a universally quantified τ-formula $\varphi_1 \wedge \cdots \wedge \varphi_k \implies \psi$, where $\varphi_i (1 \leq i \leq k)$ and ψ are atomic.
- A quasivariety is a class $\mathcal{Q}$ of τ-structures axiomatised by a set of quasiequations. Any such set Σ is called a q-base for $\mathcal{Q}$. If $\mathcal{Q} = \mathrm{Mod}(\Sigma)$ for some finite set Σ of quasiequations, $\mathcal{Q}$ is said to be finitely q-based.

1. Mal'cev characterisation $Q(\mathcal{K}) = ISPP_U(\mathcal{K})$ still holds for any class $\mathcal{K}$ of τ-structures.
2. Example: 'simple graphs' are q-based by $xRy \implies yRx$ and $xRx \implies x = y$. So, they are not exactly simple: they admit a reflexive singleton. But close enough.

Finitely q-based q-varieties of simple graphs

- We have modified the notions slightly: from universal Horn sentences/classes to quasiequations/q-varieties. The only effect is that the reflexive singleton belongs to every quasivariety.

Theorem (Caicedo,Nešetřil,Pultr)

There are precisely 4 finitely generated, finitely q-based q-varieties of simple graphs:

1. $\mathcal{D}_1 = \mathrm{Mod}\{x = y\}$ *(one-element graphs).*
2. $\mathcal{D}_2 = \mathrm{Mod}\{xRy \implies x = y\}$ *(discrete graphs).*
3. $\mathcal{D}_3 = \mathrm{Mod}\{xRy \wedge yRz \implies x = z\}$ *(single edges $\uplus$ s-tons).*
4. $\mathcal{D}_4 = \mathrm{Mod}\{xRy \wedge yRz \wedge zRw \implies xRw\}$ *(complete bipartite $\uplus$ singletons).*

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- We restricted the question to quasivarieties generated by a single finite oriented path: a digraph $G = \{v_0, \dots, v_n\}$ such that
  - $\mathrm{InDegree}(v_0) = 0, \mathrm{OutDegree}(v_0) = 1$,
  - $\mathrm{InDegree}(v_n) = 1, \mathrm{OutDegree}(v_n) = 0$, and
  - $\mathrm{InDegree}(v_i) + \mathrm{OutDegree}(v_i) = 2$, for $i \in \{1, \dots, n-1\}$.
- An oriented path can be viewed as a path in an undirected sense, where each edge is given an orientation. All oriented paths are balanced, that is, admit a homomorphism into $(\mathbb{N}, <)$.

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- We got some results, but the mess persisted. So we changed the model-theoretic notion of substructure (induced subdigraph), to the graph-theoretic (non-induced subdigraph, or ni-subdigraph).

Diagrams

- For a finite relational structure A (think digraph), take the set $X_A = \{x_a : a \in A\}$ of new variables and define $\mathrm{Diag}^+(A)$ (the positive diagram of A) to be the set of all atomic formulas φ such that $A \models \varphi$ under the canonical valuation $x_a \mapsto a$.
- Define $\mathrm{Diag}^-(A)$ (the negative diagram) to be the set of all atomic formulas φ such that $A \not\models \varphi$ under the canononical valuation.
- Define $\mathrm{Diag}^{\neq}(A)$ (the anti-equational diagram) of A to be the set of all equations ε such that $A \not\models \varepsilon$ under the canononical valuation.
- If A is a digraph, define the diagram of edges to be $\mathrm{Diag}^+_R(A) = \mathrm{Diag}^+(A) \cap \{x_a R x_b : a, b \in A\}$.

Diagrams and characteristic formulas

By finiteness of A and the fact that the language is relational, $\mathrm{Diag}^+(A)$, $\mathrm{Diag}^-(A)$, and $\mathrm{Diag}^{\neq}(A)$ are finite. Put

$$\chi_A = \forall \overline{x} : \bigwedge \mathrm{Diag}^+(A) \Rightarrow \bigvee \mathrm{Diag}^-(A),$$
$$\alpha_A = \forall \overline{x} : \bigwedge \mathrm{Diag}^+(A) \Rightarrow \bigvee \mathrm{Diag}^{\neq}(A),$$

where $\overline{x}$ is a sequence of all new variables.

Lemma

For arbitrary finite A and A' the following conditions are equivalent:

1. *χ_A and $\chi_{A'}$ are logically equivalent,*
2. *α_A and $\alpha_{A'}$ are logically equivalent,*
3. *A and A' are isomorphic.*

Equality forcing quasiequations, noninduced subdigraphs

From now on, we restrict attention to digraphs.

- A quasiequation $\varphi_1 \wedge \cdots \wedge \varphi_n \implies \psi$, where ψ is an equation, will be called an equality-forcing quasiequation or ef-quasiequation.
- An ef-quasivariety will be any quasivariety defined by a set Σ^{ef} of ef-quasiequations.
- Let S_N be the class operator of taking noninduced subdigraphs (for short: ni-digraphs).

Lemma

A class $\mathcal{K}$ is an ef-quasivariety if and only if $\mathcal{K} = IS_N PP_U(\mathcal{K})$.

- Graphs are not a sub-ef-quasivariety of digraphs.

Critical subdigraphs

- A finite digraph D is $\mathcal{Q}$-critical, for a quasivariety $\mathcal{Q}$ if D does not belong to $\mathcal{Q}$ but all its proper subdigraphs do.
- Then D is critical if it is critical relative to the quasivariety generated by its proper subdigraphs.

Lemma (Relative finite q-bases)

Let $\mathcal{Q}$ and $\mathcal{M}$ be quasivarieties of digraphs such that $\mathcal{Q} \subseteq \mathcal{M}$. Then $\mathcal{Q}$ is finitely based relative to $\mathcal{M}$ iff there is a finite number n such that for every $D \in \mathcal{M}$, we have the equivalence

$$D \in \mathcal{Q} \iff \text{every at most } n\text{-element subdigraph of } D \text{ is in } \mathcal{Q}.$$

Critical noninduced-subdigraphs

- A finite digraph D is $\mathcal{Q}$-critical, for an ef-quasivariety $\mathcal{Q}$ if D does not belong to $\mathcal{Q}$ but all its proper ni-subdigraphs belong to $\mathcal{Q}$.
- Then D is critical if it is critical relative to the ef-quasivariety generated by its proper ni-subdigraphs.

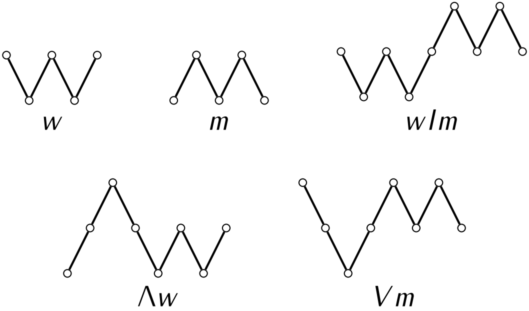
Lemma (Relative finite ef-q-bases)

Let $\mathcal{Q}$ be an ef-quasivariety of digraphs and $\mathcal{M}$ be a quasivariety of digraphs. Then $\mathcal{Q} \cap \mathcal{M}$ is finitely based relative to $\mathcal{M}$ iff there is a finite number m such that for every $D \in \mathcal{M}$ we have the equivalence

$$D \in \mathcal{Q} \iff \text{every ni-subdigraph of } D \text{ with at most } m \text{ edges is in } \mathcal{Q}.$$

Examples of directed paths

- We focus on directed paths. View the examples below as non-transitive Hasse diagrams.



- Excluding Vm, Λw, and wIm (by ef-q-identities), we are left with valleys and hills.

Some ef-quasivarieties

Let o stand for an irreflexive point, oo for a pair of irreflexive points without an edge, i for a pair $x \to y$, and ii for two pairs $x \to y, z \to u$. Write $\wedge$ for $x \to y \leftarrow z$ and v for $x \leftarrow y \to z$; write n for $x \to y \leftarrow z \to u$.

$$
\begin{tikzcd}
Q^{ef}(\wedge i) \arrow[d, dash] \arrow[dr, dash] & Q^{ef}(n) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & Q^{ef}(vi) \arrow[dl, dash] \arrow[d, dash] \\
Q^{ef}(\wedge) \arrow[dr, dash] & Q^{ef}(ii) \arrow[d, dash] & Q^{ef}(v) \arrow[dl, dash] \\
 & Q^{ef}(i) \arrow[d, dash] & \\
 & Q^{ef}(oo) \arrow[d, dash] & \\
 & Q^{ef}(\emptyset) = Q^{ef}(o) &
\end{tikzcd}
$$

And their finite ef-q-bases

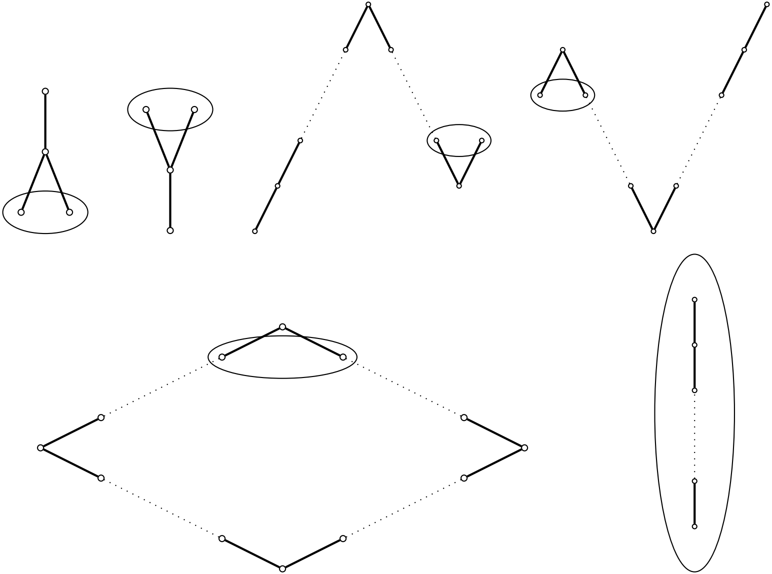
Theorem

The poset depicted on the previous slide is the set of all ef-quasivarieties generated by digraphs of height at most 1. Their respective ef-quasiequational bases are the following:

- $Q^{ef}(o) = \mathrm{Mod}\{x = y\}$,
- $Q^{ef}(oo) = \mathrm{Mod}\{x \to y \implies u = v\}$,
- $Q^{ef}(i) = \mathrm{Mod}\{\mathbf{q},\ x \to y \wedge u \to v \implies x = u \wedge y = v\}$,
- $Q^{ef}(\mathsf{N}) = \mathrm{Mod}\{\mathbf{q},\ x \to y \wedge u \to v \implies y = v\}$,
- $Q^{ef}(\mathsf{N}i) = \mathrm{Mod}\{\mathbf{q},\ x \leftarrow y \to z \leftarrow u \implies x = z\}$,
- $Q^{ef}(ii) = \mathrm{Mod}\{\mathbf{q},\ x \to y \leftarrow z \implies x = z,\ x \leftarrow y \to z \implies x = z\}$,
- $Q^{ef}(v) = \mathrm{Mod}\{\mathbf{q},\ x \to y \wedge u \to v \implies x = u\}$,
- $Q^{ef}(vi) = \mathrm{Mod}\{\mathbf{q},\ x \leftarrow y \to z \leftarrow u \implies y = u\}$,
- $Q^{ef}(n) = \mathrm{Mod}\{\mathbf{q}\}$,

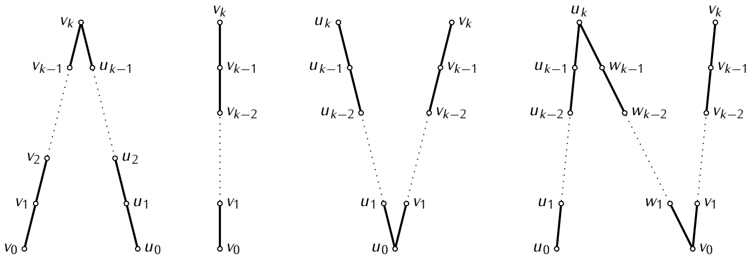
where $\mathbf{q}$ *is the ef-quasiequation* $x \to y \to z \implies u = v$, *defining height at most 1.*

Pictorial notation



Some finite ef-q-basedness

For any $k \geq 1$, let Λ_k, I_k, V_k and N_k be the paths depicted below:



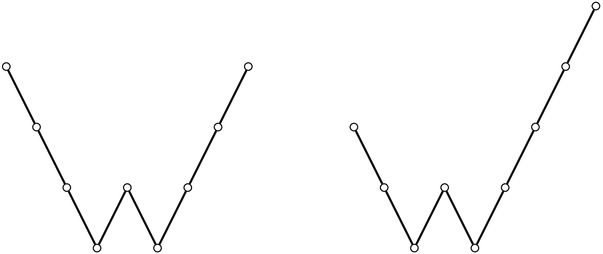
Theorem

Relative to balanced digraphs:

- $Q^{ef}(D_k)$ *is finitely ef-q-based, for* $D_k \in \{\Lambda_k, I_k, V_k, N_k\}$.

Valleys and hills

- Consider oriented paths that contain w or m but $\{wIm, \Lambda w, Vm\} \cap Q(P) = \emptyset$. We call them valleys and hills.
- Critical digraphs (even critical paths!) relative to valleys and hills may still be very complex. For consider the following valleys



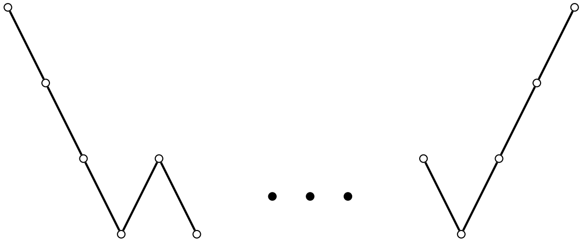
W_{33} W_{24}

Then W_{33} is finitely ef-q-based while W_{24} is not.

Valleys and hills, cont'd

- Finite ef-q-base of W_{33} has two crucial ingredients:
  1. No-flail law.
  2. Only-distant-towers law.

  … plus some less important details.
- Non-finite ef-q-basedness of W_{24} is witnessed by the infinite sequence of critical (relative to W_{24}) digraphs below:



One general theorem

- Let $\mathcal{M}_{h,w}$ be some quasivariety of graphs, with parameters h for height, and w for width.
- Assume $\mathcal{M}_{h,w}$ is big enough to contain all paths P that we ever want to consider.
- $\mathcal{M}_{h,w}$ itself may not be an ef-quasivariety!

Theorem

Let P be a finite oriented path.

1. *If P contains Vm, Λw or wIm, then $Q^{ef}(P)$ is not finitely ef-q-based.*
2. *There is a linear function $c(h,w)\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every finite oriented path P that does not contain Vm, Λw and wIm, the following are equivalent:*
   - 2.1 *$Q^{ef}(P)$ is finitely ef-q-based relative to $\mathcal{M}_{h,w}$ containing P.*
   - 2.2 *All critical digraphs relative to P have at most $c(h,w)$ vertices.*

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