Graphical Initiality for Categorical Linear Dependency

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Plan of the talk

1. Background and Overview
2. Dependent Graded Linear Categories
3. (A Sketch of) Graphical Initiality

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is. The idea is to *grade variables in types by zero* [McBride, 2016], so that they do not conflict with variables consumed by linear terms.

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3. Every initial small dependent graded linear category can model MLTT with dependent sums and products, *WF-trees*, identities and a universe.

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$$\omega_\Gamma : \Gamma^0 \to I, \qquad \kappa[p,q]_\Gamma : \Gamma^{p+q} \to \Gamma^p \otimes \Gamma^q, \qquad \epsilon_\Gamma : \Gamma^1 \xrightarrow{\sim} \Gamma,$$

$$\delta[p,q]_\Gamma : \Gamma^{pq} \xrightarrow{\sim} (\Gamma^p)^q, \qquad \iota[p] : I^p \to I,$$

$$\mu[p]_{\Gamma,\Delta} : (\Gamma \otimes \Delta)^p \to \Gamma^p \otimes \Delta^p \quad (\Gamma, \Delta \in \mathcal{M}, p, q \in \mathcal{R}),$$

such that the following diagrams commute:

Module Categories (3/3)

$$\begin{tikzcd}
\Gamma^0 \arrow[r, "\omega"] \arrow[d, "{\delta[0,r]}"'] & I \\
(\Gamma^0)^r \arrow[r, "\omega^r"'] & I^r \arrow[u, "{\iota[r]}"']
\end{tikzcd}
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$$\begin{tikzcd}
(\Gamma^{p+q})^r \arrow[r, "{\kappa[p,q]^r}"] & (\Gamma^p \otimes \Gamma^q)^r \arrow[r, "{\mu[r]}"] & (\Gamma^p)^r \otimes (\Gamma^q)^r \\
\Gamma^{pr+qr} \arrow[u, "{\delta[p+q,r]}"] \arrow[r, "{\kappa[pr,qr]}"'] & \Gamma^{pr} \otimes \Gamma^{qr} \arrow[ur, "{\delta[p,r] \otimes \delta[q,r]}"'] &
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Proposition

Every graded linear category forms a linear category.

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- Let $\mathcal{M}[0]_{\simeq}$ be the **WP-completion** of $\mathcal{M}[0]$, and $(\_)^0_{\simeq} \downarrow (\_)_{\simeq}$ the comma category of the functors $(\_)^0_{\simeq} : \mathcal{M} \to \mathcal{M}[0]_{\simeq} \leftarrow \mathcal{M}[0] : (\_)_{\simeq}$.

Thus, an object in $(\_)^0_{\simeq} \downarrow (\_)_{\simeq}$ is a morphism $A : \Omega^0 \to \Delta^0$ in $\mathcal{M}[0]_{\simeq}$, and a morphism from $A : \Omega^0 \to \Delta^0$ to $B : \Xi^0 \to \Gamma^0$ is a pair (f, ϕ) of those $f : \Omega \to \Xi$ in $\mathcal{M}$ and $\phi : \Delta^0 \to \Gamma^0$ in $\mathcal{M}[0]$ such that

$$\begin{tikzcd} \Omega^0 \arrow[r, "{[f^0]}"] \arrow[d, "A"'] & \Xi^0 \arrow[d, "B"] \\ \Delta^0 \arrow[r, "{[\phi]}"'] & \Gamma^0 \end{tikzcd} \qquad \begin{tikzcd} \Delta^0 \arrow[r, "{[\langle \phi, b\rangle^0]}"] \arrow[d, "\mathrm{id}"'] & (\Gamma^0.B)^0 \arrow[d, "B"] \\ \Delta^0 \arrow[r, "{[\phi]}"'] & \Gamma^0 \end{tikzcd} \qquad \dfrac{\Delta^0 \vdash \phi : \Gamma^0 \quad \Gamma^0 \vdash B \text{ type} \quad \Delta \vdash b : B[\phi]}{\Delta \vdash \langle \phi, b \rangle : \Gamma^0.B}$$

Dependent Graded Linear Categories (2/4)

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1. $(\_)^0_{\simeq} \downarrow (\_)_{\simeq} \xrightarrow{\mathrm{cod}} \mathcal{M}[0] \xrightarrow{(\_)_{\simeq}} \mathcal{M}[0]_{\simeq}$ *itself is* *not* *suited for dependent graded linear categories; e.g., comprehension is always strict.*

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1. The creation & initiality of $\mathcal{E}$ *canonically* yield types and terms;
2. The smallness of $\mathcal{E}$ keeps our initial graphical calculus *finitary*.

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- **Exponential** ! if so does each $\mathcal{E}_\Gamma$, and $\mathrm{dom} \circ \iota$ creates it.

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Every small dependent graded linear category has identities, top, one, dependent sums/products and exponential, and WF-trees if it is initial.

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Every fibre of a dependent module category is module, and similarly for a dependent graded linear category.

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Plan of the talk

Initiality for Linear Categories (1/3)

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Theorem ([Yamada, 2026])

*The **ℓ-calculus** forms a finitary graphical initial linear category, in which quantitative information is given inherently by **colouring**.*

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Initiality for Graded Linear Categories

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Definition (the $\gamma\ell$-calculus)

The $\boldsymbol{\gamma\ell}$**-calculus** is the extension of the ℓ-calculus with the construct $(\_)^n$ for each $n \in \mathbb{N}$ that is the restriction of ! to n *weighted* colours.

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(\perp^3_{[x]})^2 & \dashv\vdash & \perp^1_{[y]} \arrow[ll, bend right, "1"'] & \otimes & \perp^2_{[z]} \arrow[llll, bend left, "2"] & \otimes & \perp^3_{[w]} \arrow[llllll, bend right, "3"']
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$$\perp^0_{[x]} \dashv\vdash 1^{10}$$

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$$(\perp^3_{[x]})^2 \dashv\vdash \perp^1_{[y]} \otimes \perp^2_{[z]} \otimes \perp^3_{[w]}$$

$$\top^7_{[x]} \dashv\vdash \top^1_{[y]}$$

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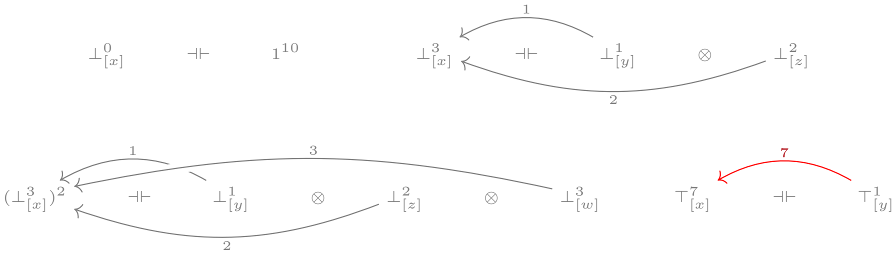

Theorem

The $\gamma\ell$-calculus forms a finitary graphical *initial* *graded linear category.*

Initiality for Dependent Graded Linear Categories

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The $\boldsymbol{\delta\gamma\ell}$**-calculus** is the extension of the $\gamma\ell$-calculus with a **universe**.

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The $\delta\gamma\ell$-calculus forms a finitary graphical initial small dependent graded linear category.

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