Some new semidirect-product-type constructions of residuated lattices

Michal Botur

Faculty of Science, Palacký University Olomouc
michal.botur@upol.cz

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- As soon as these two axioms are dropped, the known representation techniques begin to fail, or become so abstract that they say very little about the structure of the algebra itself.
- This talk reports on some progress in a project which tries to fill exactly this gap.

Our aim

- **Goal:** to introduce new product-type constructions for certain classes of residuated structures (hoops and integral residuated lattices) which behave well with respect to associativity.
- **Main representation** (finite case): $\mathbf{A} \cong \mathbf{F} \ltimes (\mathbf{A}/F)$ for every filter $F \in \mathbf{Fil}\,\mathbf{A}$.
- **Inspiration:** wreath and semidirect products of semigroups, and the Krohn–Rhodes theorem.
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If $\mathbf{A}$ and $\mathbf{B}$ are hoops and $(T_b)_{b\in|\mathbf{B}|}$ is a suitable system of transformations, we want the universe of the product to be

$$|\mathbf{A} \ltimes \mathbf{B}| = \textstyle\sum_{b\in|\mathbf{B}|} T_b(\mathbf{A}),$$

i.e. every element of the "skeleton" $\mathbf{B}$ is replaced by a transformed copy of $\mathbf{A}$.

The model: wreath products

- In a wreath product the second factor plays the role of a skeleton, and every element of the skeleton induces a transformation of the first factor.
- Associativity holds because the relevant transformations are composed in the right way (as mappings): the cascade expansion does not depend on the order in which it is performed.

Hoops

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Definition

A hoop is an algebra $\mathbf{A} = (A; \cdot, \to, 1)$ *of type* $\langle 2, 2, 0 \rangle$ *such that* $(A; \cdot, 1)$ *is a commutative monoid satisfying the identities*

(H1) $x \to x = 1$,

(H2) $(x \cdot y) \to z = x \to (y \to z)$,

(H3) $x \cdot (x \to y) = y \cdot (y \to x)$.

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The induced order is given by $x \leq y$ if and only if $x \to y = 1$, and moreover $x \wedge y = x \cdot (x \to y)$.

Equivalently, a hoop is a naturally ordered commutative monoid satisfying the adjointness property

$$x \cdot y \leq z \Longleftrightarrow x \leq y \to z.$$

Filters

If $\mathbf{A} = (A; \cdot, \to, 1)$ is a hoop, then a *filter* is a nonempty subset $F \subseteq A$ such that

(F1) if $x \in F$ and $x \leq y$, then $y \in F$,

(F2) if $x, y \in F$, then $x \cdot y \in F$.

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The set of all filters of $\mathbf{A}$ is denoted by $\mathbf{Fil}\,\mathbf{A}$. Every filter is a subalgebra (written $\mathbf{F}$) and induces the congruence

$$\theta_F = \{\langle x, y \rangle \in A^2 \mid (x \rightarrow y) \cdot (y \rightarrow x) \in F\},$$

so that we may write $\mathbf{A}/F$ for the corresponding quotient. This correspondence between filters and congruences is a lattice isomorphism.

The f-product of hoops

Definition

Let $\mathbf{A} = (A; \cdot, \to, 1)$ *and* $\mathbf{B} = (B; \cdot, \to, 1)$ *be hoops. A mapping* $f\colon A \longrightarrow B$ *is called a product morphism from* $\mathbf{A}$ *to* $\mathbf{B}$ *if it satisfies*

(pM1) $f(1) = 1$,

(pM2) $f(x) \cdot f(y) = f(x \cdot y) = f(x) \wedge f(y) = f(x \wedge y)$

for all $x, y \in A$. *(Hence* f *is monotone and* $f(A) \subseteq \mathbf{Id}\,\mathbf{B}$.*)*

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Theorem

If $\mathbf{A}$ and $\mathbf{B}$ are hoops and $f : \mathbf{B} \longrightarrow \mathbf{A}$ is a product morphism, then

$$\mathbf{A} \ltimes_f \mathbf{B} = (\textstyle\sum_{x \in B} (f(x)], \cdot, \to, (1,1)),$$

where

$(\cdot)$ $(a, x) \cdot (b, y) := (a \cdot b,\ x \cdot y)$,

$(\to)$ $(a, x) \to (b, y) := (f(x \to y) \wedge (a \to b),\ x \to y)$,

is a hoop, called the f-product of $\mathbf{A}$ and $\mathbf{B}$. Its order is inherited from the direct product: $(a, x) \leq (b, y)$ iff $a \leq b$ and $x \leq y$.

Two extreme examples

Example

For arbitrary hoops $\mathbf{A}, \mathbf{B}$ *the constant mapping* $\varepsilon \colon \mathbf{B} \longrightarrow \mathbf{A}$*,* $\varepsilon(x) = 1$*, is a product morphism, and it is the greatest one with respect to the natural order. Here* $\mathbf{A} \ltimes_\varepsilon \mathbf{B} = \mathbf{A} \times \mathbf{B}$.

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Example

If $\mathbf{A}$ *has a least element* 0*, then* $\sigma\colon \mathbf{B} \longrightarrow \mathbf{A}$ *given by* $\sigma(1) = 1$ *and* $\sigma(x) = 0$ *for* $x \neq 1$ *is the least product morphism, and* $\mathbf{A} \ltimes_{\sigma} \mathbf{B} = \mathbf{B} \oplus \mathbf{A}$ *(the ordinal sum).*

Associated pairs of product morphisms

Definition

Let $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be hoops. A pair* (f, g) *is a left-associated pair of product morphisms with respect to* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *if*

$$f : \mathbf{B} \longrightarrow \mathbf{A} \qquad \textit{and} \qquad g : \mathbf{C} \longrightarrow \mathbf{A} \ltimes_f \mathbf{B}$$

are product morphisms; we therefore have the hoop $(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C}$*.*

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Similarly, (f, g) *is a right-associated pair with respect to* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *if*

$$g\colon \mathbf{C} \longrightarrow \mathbf{B} \qquad \textit{and} \qquad f\colon \mathbf{B} \ltimes_g \mathbf{C} \longrightarrow \mathbf{A}$$

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are product morphisms; we therefore have the hoop $\mathbf{A} \ltimes_f (\mathbf{B} \ltimes_g \mathbf{C})$.

In a left-associated pair, g maps into the algebra $\mathbf{A} \ltimes_f \mathbf{B}$ that has already been built, so it also records how $\mathbf{C}$ acts on $\mathbf{A}$. The correspondences α and β below do nothing but rearrange this information.

Associativity up to isomorphism

Theorem

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be hoops.

(i) *If (f, g) is left-associated and we write $g(c) = (g_1(c), g_2(c))$, then*

$$\overline{g}(c) = g_2(c), \qquad \overline{f}(b, c) = f(b) \wedge g_1(c)$$

define a right-associated pair $\alpha(f, g) = (\overline{f}, \overline{g})$.

(ii) *If (f, g) is right-associated, then*

$$\overline{f}(b) = f(b, 1), \qquad \overline{g}(c) = \big(f(g(c), c),\, g(c)\big)$$

define a left-associated pair $\beta(f, g) = (\overline{f}, \overline{g})$.

(iii) *α and β are mutually inverse bijections, and if $\alpha(f, g) = (\overline{f}, \overline{g})$, then*

$$(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C} \cong \mathbf{A} \ltimes_{\overline{f}} (\mathbf{B} \ltimes_{\overline{g}} \mathbf{C}).$$

Decomposition of finite hoops

Lemma

Let $\mathbf{A}$ *be a finite hoop and* $F \in \mathbf{Fil}\,\mathbf{A}$*. Denote by* t_X *the top element and by* l_X *the least element of the class* $X \in \mathbf{A}/F$*, and write* l *instead of* l_F*. Then, for all* $X, Y \in \mathbf{A}/F$ *and all* $a \in F$*:*

(i) $X \leq Y$ *if and only if* $t_X \leq t_Y$*,*

(ii) $t_X \wedge t_Y = t_{X \wedge Y}$ *and* $t_X \to t_Y = t_{X \to Y}$*,*

(iii) $t_X \cdot t_{X \to Y} = t_{X \wedge Y}$*,*

(iv) $a \to t_X = t_X$ *and* $t_X \cdot a = t_X \wedge a$*,*

(v) $t_X \cdot l = t_X \wedge l = l_X$*.*

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(i) $X \leq Y$ *if and only if* $t_X \leq t_Y$,

(ii) $t_X \wedge t_Y = t_{X \wedge Y}$ *and* $t_X \to t_Y = t_{X \to Y}$,

(iii) $t_X \cdot t_{X \to Y} = t_{X \wedge Y}$,

(iv) $a \to t_X = t_X$ *and* $t_X \cdot a = t_X \wedge a$,

(v) $t_X \cdot l = t_X \wedge l = l_X$.

Hence every class X is, via $a \mapsto a \vee l$, in bijection with the principal downset $(t_X \vee l]$ of the filter $\mathbf{F}$.

The decomposition theorem

Theorem

If $\mathbf{A}$ is a finite hoop and $F \in \mathbf{Fil}\,\mathbf{A}$, then $\psi\colon \mathbf{A}/F \longrightarrow \mathbf{F}$ defined by $\psi(X) = t_X \vee l$ is a product morphism and, moreover, $\mathbf{A} \cong \mathbf{F} \ltimes_\psi (\mathbf{A}/F)$.

The isomorphism is given by $\Omega(a) = (a \vee l,\ a/F)$, with $\Omega^{-1}(a, X) = a \wedge t_X$; the two pictures shown next are decompositions of exactly this kind.

Finite hoops as products of MV-chains

A hoop $\mathbf{A}$ is *irreducible* if $\mathbf{A} \cong \mathbf{B} \ltimes_f \mathbf{C}$ implies that $\mathbf{B}$ or $\mathbf{C}$ is trivial.

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A finite hoop is irreducible if and only if it is simple.

Theorem (Blok, Ferreirim)

Simple finite hoops are precisely the finite MV-chains.

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Corollary

For every finite hoop $\mathbf{A}$ *there exist finite MV-chains* $\mathbf{M}_{i_1}, \ldots, \mathbf{M}_{i_n}$ *and product morphisms* $f_1, \ldots, f_{n-1}$ *such that*

$$\mathbf{A} \cong \mathbf{M}_{i_1} \ltimes_{f_1} (\mathbf{M}_{i_2} \ltimes_{f_2} (\cdots \ltimes_{f_{n-2}} (\mathbf{M}_{i_{n-1}} \ltimes_{f_{n-1}} \mathbf{M}_{i_n}) \ldots)),$$

and, by associativity, also with the opposite bracketing.

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The same conclusion is known via poset products: every finite GBL-algebra, hence every finite hoop, is a poset product of finite MV-chains [Di Nola–Lettieri 2003; Jipsen–Montagna 2009]. New here is a *binary, associative* product — and that is what survives when divisibility is dropped.

Products with an MV-chain factor

Let $\mathbf{A}$ be a finite hoop and let $\mathbf{M}$ be a finite MV-chain, so that $\mathbf{Id}\,\mathbf{M} = \{0,1\}$. For an idempotent $f \in \mathbf{Id}\,\mathbf{A}$, both

$$\psi_f \colon \mathbf{A} \longrightarrow \mathbf{M}, \quad \psi_f(x) = 1 \iff f \leq x,$$
$$\psi_f \colon \mathbf{M} \longrightarrow \mathbf{A}, \quad \psi_f(1) = 1 \ \text{ and } \ \psi_f(x) = f \text{ for } x \neq 1,$$

are product morphisms.

Theorem

$\psi \mapsto \min \psi^{-1}(1)$ and $\psi \mapsto \psi(0)$ are bijections between $\mathbf{Id}\,\mathbf{A}$ and the set of all product morphisms $\mathbf{A} \longrightarrow \mathbf{M}$, resp. $\mathbf{M} \longrightarrow \mathbf{A}$.

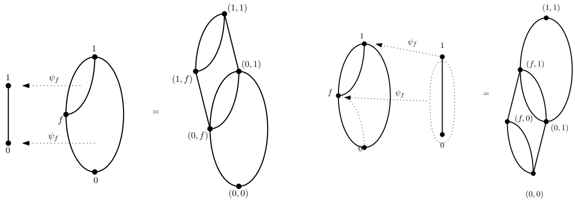

Figure 1: $\mathbf{M} \ltimes_{\psi_f} \mathbf{A}$ and $\mathbf{A} \ltimes_{\psi_f} \mathbf{M}$.

The f-product as a split extension

The trivial hoop is a zero object, so kernels are exactly filters. Now $\pi_2 \colon \mathbf{A} \ltimes_f \mathbf{B} \longrightarrow \mathbf{B}$ has kernel $\pi_2^{-1}(1) = \{(a, 1) \mid a \in A\} \cong \mathbf{A}$, and it *splits*:

$$s \colon \mathbf{B} \longrightarrow \mathbf{A} \ltimes_f \mathbf{B}, \qquad s(x) = (f(x), x), \qquad \pi_2 \circ s = \mathrm{id}_{\mathbf{B}},$$

since $s(x) \cdot s(y) = (f(x \cdot y), x \cdot y)$ and
$s(x) \to s(y) = (f(x \to y), x \to y)$, using
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In a finite hoop every quotient splits canonically.

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So $\ltimes_f$ is a split extension in the sense of Bourn–Janelidze — with the acting maps being nuclei rather than automorphisms. The corresponding picture for exact sequences is developed in the appendix of the paper.

Generalisation I: nuclei

From now on **A**, **B**, **C** are arbitrary **integral residuated lattices**: neither commutativity nor divisibility is assumed.

A closure operator γ on **A** is a *nucleus* if $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$.

Why nuclei? The fibre $(f(x)]$ is the image of the nucleus $a \mapsto f(x) \to a$; noncommutatively, no such generator need exist.

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Lemma

*If a closure operator γ on **A** satisfies*

$$\gamma(x)\backslash\gamma(y) = \gamma(x\backslash y)\ (SN1) \quad \textit{or} \quad \gamma(x)/\gamma(y) = \gamma(x/y)\ (SN2),$$

then it is a nucleus and $\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$.

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A nucleus satisfying both (SN1) and (SN2) is a *strong nucleus*, and $\mathrm{SN}\,\mathbf{A}$ denotes the set of all strong nuclei on **A**. Strong nuclei commute, $\alpha \circ \beta \in \mathrm{SN}\,\mathbf{A}$, and $(\mathrm{SN}\,\mathbf{A}; \circ, I_A)$ is a semilattice in which $\alpha \lhd \beta$ if and only if $\alpha A \subseteq \beta A$.

The product via nuclei

Definition (Jan Kühr)

If $\mathbf{A}$ *and* $\mathbf{B}$ *are integral residuated lattices, then a mapping* $\boldsymbol{\alpha}\colon B \longrightarrow \mathrm{SN}\,\mathbf{A}$ *satisfying*

$$\boldsymbol{\alpha}[1] = I_A \qquad \textit{and} \qquad \boldsymbol{\alpha}[x] \circ \boldsymbol{\alpha}[y] = \boldsymbol{\alpha}[x \wedge y] = \boldsymbol{\alpha}[x \cdot y] \quad (x, y \in B)$$

is a nuclei-morphism from $\mathbf{B}$ *to* $\mathbf{A}$*, written* $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$*.*

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Theorem

If $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$*, then the algebra* $\mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}$ *with universe* $\sum_{x \in B} \boldsymbol{\alpha}[x]A$*, unit* $(1,1)$ *and*

$$\begin{aligned}(a_1, b_1) \cdot (a_2, b_2) &= (\boldsymbol{\alpha}[b_1 \cdot b_2](a_1 \cdot a_2)\,,\ b_1 \cdot b_2),\\ (a_1, b_1)\backslash(a_2, b_2) &= (a_1 \backslash \boldsymbol{\alpha}[b_1](a_2), b_1 \backslash b_2),\\ (a_2, b_2)/(a_1, b_1) &= (\boldsymbol{\alpha}[b_1](a_2)/a_1, b_2/b_1),\end{aligned}$$

is again an integral residuated lattice; its lattice operations are those of the nucleus images.

Associativity of the product via nuclei

Theorem

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be integral residuated lattices.*

(i) *For all* $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$ *and* $\boldsymbol{\beta}\colon \mathbf{C} \curvearrowright \mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}$ *there exist*

$$\overline{\boldsymbol{\beta}}\colon \mathbf{C} \curvearrowright \mathbf{B} \qquad \textit{and} \qquad \overline{\boldsymbol{\alpha}}\colon \mathbf{B} \ltimes_{\overline{\boldsymbol{\beta}}} \mathbf{C} \curvearrowright \mathbf{A}$$

such that $(\mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}) \ltimes_{\boldsymbol{\beta}} \mathbf{C} \cong \mathbf{A} \ltimes_{\overline{\boldsymbol{\alpha}}} (\mathbf{B} \ltimes_{\overline{\boldsymbol{\beta}}} \mathbf{C})$.

(ii) *For all* $\boldsymbol{\beta}\colon \mathbf{C} \curvearrowright \mathbf{B}$ *and* $\boldsymbol{\alpha}\colon \mathbf{B} \ltimes_{\boldsymbol{\beta}} \mathbf{C} \curvearrowright \mathbf{A}$ *there exist*

$$\widehat{\boldsymbol{\alpha}}\colon \mathbf{B} \curvearrowright \mathbf{A} \qquad \textit{and} \qquad \widehat{\boldsymbol{\beta}}\colon \mathbf{C} \curvearrowright \mathbf{A} \ltimes_{\widehat{\boldsymbol{\alpha}}} \mathbf{B}$$

such that $\mathbf{A} \ltimes_{\boldsymbol{\alpha}} (\mathbf{B} \ltimes_{\boldsymbol{\beta}} \mathbf{C}) \cong (\mathbf{A} \ltimes_{\widehat{\boldsymbol{\alpha}}} \mathbf{B}) \ltimes_{\widehat{\boldsymbol{\beta}}} \mathbf{C}$.

Moreover, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mapsto (\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ *and* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mapsto (\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ *are mutually inverse bijections.*

The transfer formulas, and the relation to f-products

The nuclei-morphisms above are given explicitly by

$$\overline{\boldsymbol{\beta}}[c](b) = (\pi_2 \circ \boldsymbol{\beta}[c])(1, b), \qquad \overline{\boldsymbol{\alpha}}[b, c](a) = (\boldsymbol{\alpha}[b] \circ \boldsymbol{\beta}_1[c])(a),$$

$$\widehat{\boldsymbol{\alpha}}[b](a) = \boldsymbol{\alpha}[b, 1](a), \qquad \widehat{\boldsymbol{\beta}}[c](a, b) = (\boldsymbol{\alpha}[\boldsymbol{\beta}[c](b), c](a), \boldsymbol{\beta}[c](b)),$$

where $\boldsymbol{\beta}_1[c](a) = (\pi_1 \circ \boldsymbol{\beta}[c])(a, 1)$.

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In the finite commutative case the two constructions correspond to each other one-to-one:

$$\boldsymbol{\alpha}_\phi[x](a) = \phi(x) \to a, \qquad \phi_{\boldsymbol{\alpha}}(x) = \min\{a \in A \mid \boldsymbol{\alpha}[x](a) = 1\}.$$

Generalisation II: product morphisms

Definition

A mapping $\phi\colon B \to A$ *between integral residuated lattices is a product morphism if*

$$\phi(1) = 1, \quad \phi(x) \wedge \phi(y) = \phi(x \wedge y), \quad \phi(x) \cdot \phi(y) \leq \phi(x \cdot y).$$

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Theorem

For a product morphism $\phi\colon \mathbf{B} \to \mathbf{A}$ *the algebra*

$$\mathbf{A} \ltimes_\phi \mathbf{B} = (\textstyle\sum_{x \in B} (\phi(x)], \wedge, \vee, \cdot, \backslash, /, (1,1)),$$

with componentwise $\wedge$, $\vee$, $\cdot$ *and*

$$(a, x) \bullet (b, y) = ((a \bullet b) \wedge \phi(x \bullet y),\ x \bullet y) \qquad \text{for } \bullet \in \{\backslash, /\},$$

is a residuated lattice.

Associativity for integral residuated lattices

Theorem

All mappings below are product morphisms between integral residuated lattices.

(i) *For all* $\alpha\colon \mathbf{B} \to \mathbf{A}$ *and* $\beta\colon \mathbf{C} \to \mathbf{A} \ltimes_\alpha \mathbf{B}$ *there exist*

$$\rho\beta\colon \mathbf{C} \to \mathbf{B} \qquad \textit{and} \qquad \rho\alpha\colon \mathbf{B} \ltimes_{\rho\beta} \mathbf{C} \to \mathbf{A}$$

such that $(\mathbf{A} \ltimes_\alpha \mathbf{B}) \ltimes_\beta \mathbf{C} \cong \mathbf{A} \ltimes_{\rho\alpha} (\mathbf{B} \ltimes_{\rho\beta} \mathbf{C})$.

(ii) *For all* $\beta\colon \mathbf{C} \to \mathbf{B}$ *and* $\alpha\colon \mathbf{B} \ltimes_\beta \mathbf{C} \to \mathbf{A}$ *there exist*

$$\lambda\alpha\colon \mathbf{B} \to \mathbf{A} \qquad \textit{and} \qquad \lambda\beta\colon \mathbf{C} \to \mathbf{A} \ltimes_{\lambda\alpha} \mathbf{B}$$

such that $\mathbf{A} \ltimes_\alpha (\mathbf{B} \ltimes_\beta \mathbf{C}) \cong (\mathbf{A} \ltimes_{\lambda\alpha} \mathbf{B}) \ltimes_{\lambda\beta} \mathbf{C}$.

Here ρ *and* λ *are mutually inverse bijections, given by the very same formulas as in the hoop case:* $\rho\beta(c) = \beta_2(c)$, $\rho\alpha(b, c) = \alpha(b) \wedge \beta_1(c)$, $\lambda\alpha(b) = \alpha(b, 1)$, $\lambda\beta(c) = (\alpha(\beta(c), c), \beta(c))$.

Decomposition in the split case

Definition

Writing $\mathrm{Norm}(\mathbf{A}) = \mathrm{Cent}(\mathbf{A}) \cap \mathrm{Id}(\mathbf{A})$, *we call a residuated lattice* $\mathbf{A}$ *split if it is finite, distributive and integral, and if* $x \cdot y = x \wedge y$ *for all* $x \in \mathrm{Norm}(\mathbf{A})$ *and* $y \in \mathbf{A}$. *The class of all such algebras is denoted by* $\mathbb{SPL}$.

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Lemma

Let $\mathbf{A}$ *be a finite integral residuated lattice,* $F \in \mathrm{Filt}(\mathbf{A})$, *and let* t_X, l_X *be the top and the bottom element of* $X \in \mathbf{A}/F$; *put* $l = l_F$. *Then*

1. $X \leq Y$ *iff* $t_X \leq t_Y$ *iff* $l_X \leq l_Y$,
2. $t_X \wedge t_Y = t_{X \wedge Y}$, $t_X \setminus t_Y = t_{X \setminus Y}$ *and* $t_X \,/\, t_Y = t_{X / Y}$,
3. *if* $\mathbf{A} \in \mathbb{SPL}$, *then* $t_X \wedge l = l_X$.

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2. $t_X \wedge t_Y = t_{X \wedge Y}$, $t_X \setminus t_Y = t_{X \setminus Y}$ *and* $t_X / t_Y = t_{X/Y}$,
3. *if* $\mathbf{A} \in \mathbb{SPL}$, *then* $t_X \wedge l = l_X$.

Theorem

If $\mathbf{A} \in \mathbb{SPL}$ *and* $F \in \mathrm{Filt}(\mathbf{A})$, *then* $\phi(X) = t_X \vee l$ *defines a product morphism* $\phi\colon \mathbf{A}/F \to \mathbf{F}$ *and* $\mathbf{A} \cong \mathbf{F} \ltimes_{\phi} (\mathbf{A}/F)$.

Thank you for your attention!

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