Partially-order Płonka sums of po-algebras

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Introduction

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Płonka sums $\equiv$ glue together similar algebras.

Key for:

- Regularization of strongly irregular varieties;
- Logics of variable inclusion;
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Recently:

- Commutative idempotent involutive residuated lattices;
- Locally integral involutive semigroups;
- Residuated po-monoids.

Introduction

Goal

Comprehensive framework for Płonka sums of **residuated structures**.

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Main application

Płonka style construction (and deconstruction) of **residuated lattices from** (into) **residuated lattices**:

1. Preservation of residuation;
2. Extension of lattice operations.

Płonka sums

Semilattice directed system

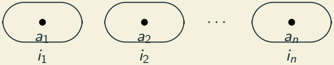
Definition

A **semilattice directed system** of τ-algebras $\mathbb{A}$ is a functor $\Phi : \mathbf{I} \to \mathsf{Alg}_\tau$.

- $\Phi(i) = \mathbf{A}_i$ **fibers**;
- $\Phi(i \leq j) = \varphi_{ij}$ **transition homomorphisms**.

Płonka sums

A **Płonka sum** over such a semilattice directed system is a τ-**algebra A** with **universe** $A = \bigsqcup_{i \in I} A_i$ such that the **operations are computed as follows** $(f \in \tau_n)$:



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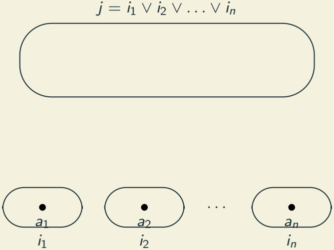

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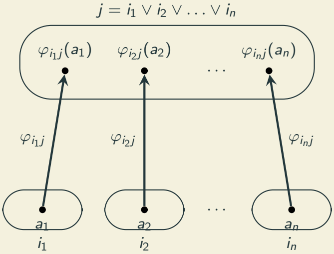

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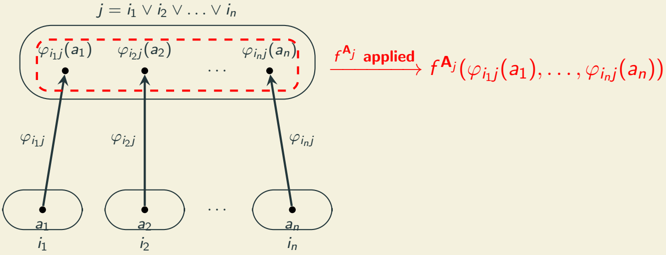

Residuated structures

Residuated maps

$(P, \leq_P)$ $(Q, \leq_Q)$

Residuated maps

$$\begin{tikzcd}(P, \leq_P) \arrow[rr, bend left, "\varphi"] & & (Q, \leq_Q)\end{tikzcd}$$

Residuated maps

$$
\begin{tikzcd}
(P, \leq_P) \arrow[r, bend left, "\varphi"] & (Q, \leq_Q) \arrow[l, bend left, "\psi"]
\end{tikzcd}
$$

$$\varphi(x) \leq_Q y \iff x \leq_P \psi(y)$$

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In particular

$$\varphi\left(\bigvee S\right) = \bigvee \varphi(S), \qquad \psi\left(\bigwedge S\right) = \bigwedge \psi(S)$$

Fully-residuated systems

Definition

A **fully-residuated system** $(f, g_1, \ldots, g_n)$ on a poset $\mathbf{P}$ is a (n+1)-tuple of n-ary operations on P such that the unary restriction $f(\ldots, -_i, \ldots)$ is residuated with residual $g_i(\ldots, -_i, \ldots)$, for each coordinate i.

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Examples

- $(n = 1)$: residuated endomorphism;
- $(n = 2)$: $\mathbf{S} = (S, \cdot^{\mathbf{S}}, \backslash^{\mathbf{S}}, / \leq_{\mathbf{S}})$ residuated po-semigroup, than $(\cdot^{\mathbf{S}}, \backslash^{\mathbf{S}}, /^{\mathbf{S}})$ is a fully residuated system on $(S, \leq_{\mathbf{S}})$.

po-Płonka sums

Po-Algebra Signatures

Definition

A **po-algebra signature** is a pair $(\mathcal{F}, \tau)$ where:

- $\mathcal{F}$ is a set of **functional symbols**;
- τ assigns a **type** (**distribution** and **monotonicity**) to each symbol:

$$\tau : \mathcal{F} \to \{\vee, \wedge\} \times \bigcup_{n=0}^{\infty} \{+, -\}^n, \quad f \mapsto (h_f, t_f)$$

Po-algebras

Definition

A **po-algebra A in the po-signature** $(\mathcal{F}, \tau)$ is a tuple $(A, F, \leq^{\mathbf{A}})$ such that:

- $(A, \leq^{\mathbf{A}})$ is a **non-empty poset**;
- $F = \{f^{\mathbf{A}}\}_{f \in \mathcal{F}}$ is a **set of operations** on A of **arity** $|t_f|$ that behave as follows:

Type of distribution	***i*-th Coordinate**	***i*-th Coordinate**
any	$+$	monotone
any	$-$	antitone
$\vee$	$+$	$\vee \to \vee$
$\vee$	$-$	$\wedge \to \vee$
$\wedge$	*dually*	

Po-algebras: examples

Example 1: Residuated po-semigroups

$\cdot \longmapsto (\vee, +, +), \qquad \backslash \longmapsto (\wedge,\ -,\ +),$
$/ \longmapsto (\wedge,\ +,\ -)$

Example 2: Bounded distributive lattices

$\vee \longmapsto (\wedge, +, +), \qquad \wedge \longmapsto (\vee, +, +)$

Example 3: Fully-residuated structures

$\mathbf{A} = (A, \{f^{\mathbf{A}}, g_1^{\mathbf{A}}, \ldots, g_n^{\mathbf{A}}\}, \leq_{\mathbf{A}})$

$$f \mapsto (\vee, +, +, \ldots, +)$$

$$g_1 \mapsto (\wedge, -, +, \ldots, +), \quad \ldots, \quad g_n \mapsto (\wedge, +, +, \ldots, -)$$

Po-metamorphisms

Definition

A **po-metamorphism** between po-algebras **A** and **B** over the same po-signature is a pair of monotone maps $\psi, \varphi : \mathbf{A} \to \mathbf{B}$ such that $\psi \leq^{\mathbf{B}} \varphi$, satisfying:

$$\varphi(f^{\mathbf{A}}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\varphi^{\tau_1}(x_1), \ldots, \varphi^{\tau_n}(x_n)) \quad (\tau(f)_0 = \vee)$$
$$\psi(g^{\mathbf{A}}(x_1, \ldots, x_n)) = g^{\mathbf{B}}(\psi^{\tau_1}(x_1), \ldots, \psi^{\tau_n}(x_n)) \quad (\tau(g)_0 = \wedge)$$

The maps **switch behavior** depending on the coordinate sign $\tau_i \in \{+, -\}$:

$\vee$-**distributing operations** (f)	$\wedge$-**distributing operations** (g)
$\varphi^+ = \varphi \qquad \varphi^- = \psi$	$\psi^+ = \psi \qquad \psi^- = \varphi$

Definition

A **semilattice directed system of po-algebras and po-metamorphisms** is a functor $\Phi : \mathbf{I} \to \mathrm{Alg}_{\tau}^{\leq}$.

Let $\psi_{ij}, \varphi_{ij} = \Phi(i \leq j) : \mathbf{A}_i \to \mathbf{A}_j$. A **po-Płonka sum** over Φ is a first order structure $\int_{\mathbf{I}} \Phi := (\mathbf{A}, \{f^{\mathbf{A}}\}_{f \in \mathcal{F}}, \leq_{\mathbf{A}})$ such that:

- $A = \bigsqcup_I A_i$
- $(a, i_1) \leq^{\mathbf{A}} (b, i_2) \iff \varphi_{i_1 j}(a) \leq^{\mathbf{A}_j} \psi_{i_2 j}(b), \quad j = i_1 \vee^{\mathbf{I}} i_2$

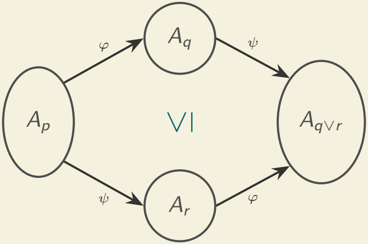
and we have $(\tau(f)_0 = \vee, \quad \tau(g)_0 = \wedge)$

$$f^{\mathbf{A}}((a_1, i_1), \ldots, (a_n, i_n)) = f^{\mathbf{A}_k}(\varphi_{i_1 k}^{\tau_1}(a_1), \ldots, \varphi_{i_n k}^{\tau_n}(a_n)),$$
$$g^{\mathbf{A}}((a_1, i_1), \ldots, (a_n, i_n)) = g^{\mathbf{A}_k}(\psi_{i_1 k}^{\tau_1}(a_1), \ldots, \psi_{i_n k}^{\tau_n}(a_n)),$$

$k := i_1 \vee \ldots \vee i_n$.

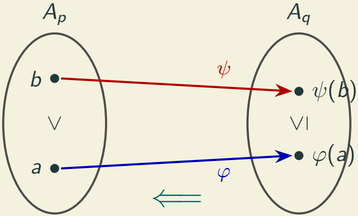
Two properties are necessary:

(S1)



$p < q, r$

(S2)



$p < q$

po-Płonka sums

Theorem

Let Φ be a semilattice directed system of po-algebras and po-metamorphisms with po-Płonka sum $\mathbf{A}$. Then:

1. The binary relation $\leq^{\mathbf{A}}$ is a **partial order** iff (S_1) and (S_2) hold (Bonzio et al., 2024);
2. **residuation is preserved**: $(f, g_1, \ldots, g_n)$ term-defined fully-residuated system on each fiber, then $(f, g_1, \ldots, g_n)$ fully-residuated system on $(\mathbf{A}, \leq_{\mathbf{A}})$;
3. if each φ and ψ is $\vee$-preserving and $\wedge$-preserving, respectively, then $\mathbf{A}$ is a po-algebra over the same po-signature of the fibers.

Remark

In a po-Płonka sum:

- Inequalities following from (generalized) residuation law are preserved;
- An identity $\alpha \approx \beta$ is valid iff if it is a regular identity (i.e. $\mathit{Var}(\alpha) = \mathit{Var}(\beta)$) satisfied in each fiber.

Step-by-step (po-)Płonka sums

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Main Idea

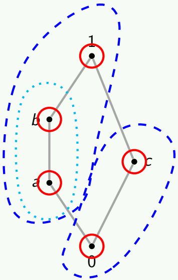
A join-semilattice is **Płonka 2-dismantlable** if it can be **recursively decomposed into 2-fiber Płonka sums** until the trivial components are reached.

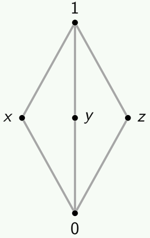
Step-by-step (po-)Płonka sums

Main Idea

A join-semilattice is **Płonka 2-dismantlable** if it can be **recursively decomposed into 2-fiber Płonka sums** until the trivial components are reached.

N_5 is Płonka 2-dismantlable, M_3 is not





Step-by-step (po-)Płonka sums

Proposition

A join-semilattice $\mathbf{J}$ is a Płonka sum over a 2-fiber semilattice direct system $\mathbb{I} = (\{\mathbf{I}, \mathbf{F}\}, \{1 < 2\}, \varphi_{12})$ of disjoint join-semilattices if and only if:

- $\mathbf{I}$ is an **ideal** of $\mathbf{J}$;
- $\mathbf{F}$ is a **retract** of $\mathbf{J}$ such that F is an **order filter** of $\mathbf{J}$.

Proposition

Let $\mathbf{J}$ be a join-semilattice and $\mathbf{F}$ a subsemilattice of $\mathbf{J}$ such that F is an order filter of $\mathbf{J}$. The following are equivalent:

(i) $\mathbf{F}$ is a **retract** of $\mathbf{J}$;

(ii) $\min(\uparrow x \cap F)$ exists for every $x \in J$.

Step-by-step (po-)Płonka sums

Remark

- Splitting pair (a, b) of $\mathbf{J} \longrightarrow$ 2-fiber Płonka sum decomposition,
- The previous is a 1-to-1 correspondence if $\mathbf{J}$ is a **finite lattice** (i.e. a **finite join-semilattice with zero**).

Step-by-step (po-)Płonka sums

Theorem (Associativity for po-Płonka sums)

- $\Phi : \mathbf{I} \to \mathsf{Alg}^{\leq}_{\tau}$ semilattice directed system of po-Algebras and metamorphisms satisfying $(S_1), (S_2)$;
- $\theta : \mathbf{J} \to \mathsf{Jslat}$ semilattice directed system of join-semilattices such that $\mathbf{I} = P\text{ł}(\theta)$.

Let $\Phi' : \mathbf{J} \to \mathsf{Alg}^{\leq}_{\tau}$, where $\Phi'(j) = \int_{\theta(j)} \Phi$, while $\Phi'(j \leq k)$ is the pair of parallel monotone maps

$$\bigcup_{i \in I_j} \varphi_{i\theta_{jk}(i)} \quad \text{and} \quad \bigcup_{i \in I_j} \psi_{i\theta_{jk}(i)}.$$

Then Φ' satisfies $(S_1), (S_2)$ and

$$\int_{\mathbf{I}} \Phi = \int_{\mathbf{J}} \Phi'.$$

Lattices from poset Płonka sums

Lower bounded and upper bounded maps

Definition

Let A and B be posets.

- An order-preserving map $\varphi : A \to B$ is **upper bounded** if for every $b \in B$, $\varphi^{-1}(\downarrow b)$ is empty or has a greatest element $\alpha_\varphi(b)$.
- An order-preserving map $\psi : A \to B$ is **lower bounded** if for every $b \in B$, $\psi^{-1}(\uparrow b)$ is empty or has a least element $\beta_\psi(b)$;

Lattices from two-fiber po-Płonka sums

Theorem

Suppose A and B are lattices, $\psi, \varphi : A \to B$ are a **lower bounded** map and an **upper bounded** map, respectively. For each $x \in A$ and $y \in B$, define the partial maps

$$x \wedge^{\varphi} y := x \wedge^{A} \alpha_{\varphi}(y), \qquad x \wedge^{\psi} y := \psi(x) \wedge^{B} y;$$

$$x \vee^{\psi} y := x \vee^{A} \beta_{\psi}(y), \qquad x \vee^{\varphi} y := \varphi(x) \vee^{B} y.$$

Then the poset po-Płonka sum of A and B is a **lattice extending A and B** if and only if the following conditions hold:

(i) (S_1) and (S_2) hold $(\psi < \varphi)$;

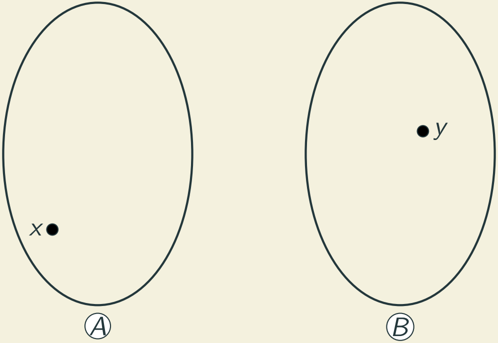
(ii) φ is **join-preserving** and ψ is **meet-preserving**;

(iii) $x \wedge^{\varphi} y$ and $x \wedge^{\psi} y$ **are comparable in** C;
$x \vee^{\psi} y$ and $x \vee^{\varphi} y$ **are comparable in** C,

if $\alpha_{\varphi}(y)$ and $\beta_{\psi}(y)$ are both defined.

Focusing on $\vee$

- A**,** B **lattices;** $\psi, \varphi : A \to B$ **parallel monotone maps**;
- $\psi < \varphi$**;** φ $\vee$**-preserving;** ψ **lower-bounded**



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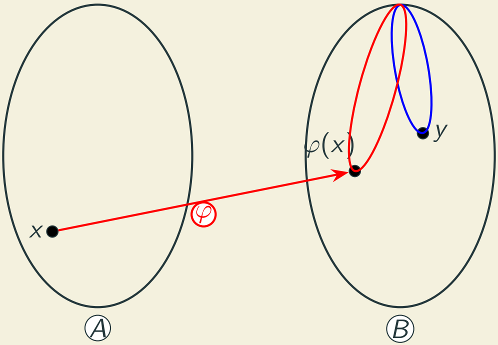

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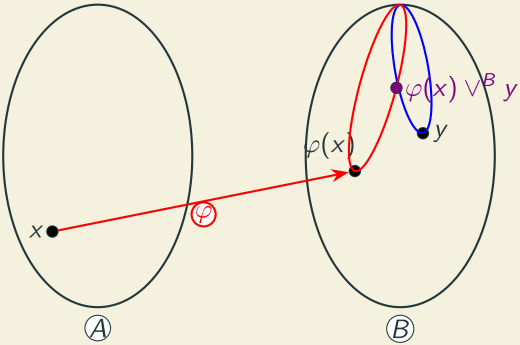

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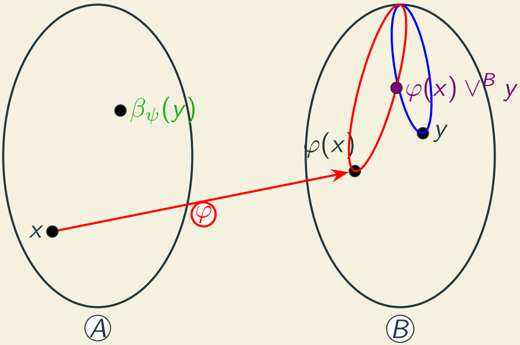

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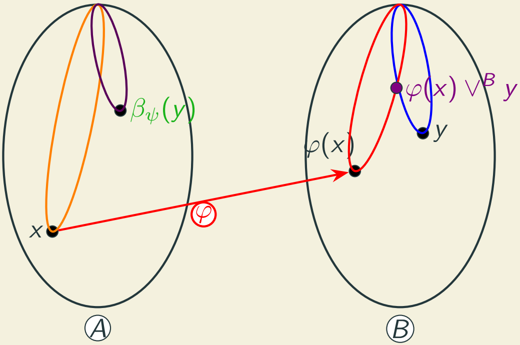

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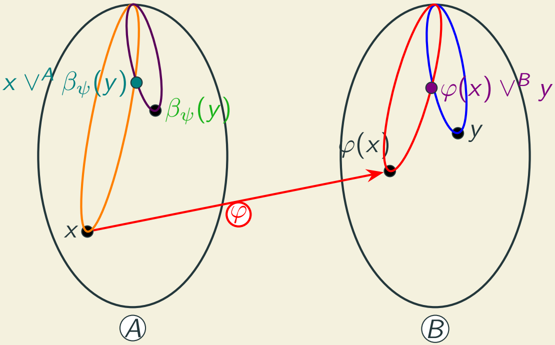

Thank you!

Questions? Remarks?

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