Product Łukasiewicz Unbound Logic

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Introduction

Goal 1

To provide generalization both **Product Łukasiewicz Logic (PŁ')** and **Łukasiewicz Unbound Logic (Łu)** establish a corresponding completeness theorem.

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- Abelian logic, Łu and PŁ$'$
- ℓ-rings, f-rings and integral domains
- Product Łukasiewicz Unbound Logic
- Translation of PŁ$'$ into PŁu'

Abelian lattice ordered groups (Abelian ℓ-groups)

Definition

We say that $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0 \rangle$ *is an Abelian lattice ordered group (ℓ-group) if* $\langle A, +, -, 0 \rangle$ *is an Abelian group,* $\langle A, \vee, \wedge \rangle$ *is a lattice and* $\mathbf{A}$ *satisfies the monotonicity condition, that means* $x \leq y$ *implies* $x + z \leq y + z$.

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We say that $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0 \rangle$ *is an Abelian lattice ordered group (*ℓ*-group) if* $\langle A, +, -, 0 \rangle$ *is an Abelian group,* $\langle A, \vee, \wedge \rangle$ *is a lattice and* $\mathbf{A}$ *satisfies the monotonicity condition, that means* $x \leq y$ *implies* $x + z \leq y + z$*.*

Definition

Let $\boldsymbol{A}$ *be an Abelian* ℓ*-group. A consecution* $\Gamma \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*,* $\Gamma \vDash_{\mathbf{A}} \varphi$ *in symbols, if for every* $\mathbf{A}$*-evaluation* e *it holds that, whenever* $e(\psi) \geq 0$ *for every* $\psi \in \Gamma$*, then also* $e(\varphi) \geq 0$*.*

Abelian logic (Ab)

(sf)	$(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$	suffixing
(e)	$(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$	exchange
(id)	$\varphi \to \varphi$	identity
(rel)	$((\varphi \to \psi) \to \psi) \to \varphi$	relativity axiom
(ub_1)	$\varphi \to \varphi \vee \psi$	upper bound
(ub_2)	$\psi \to \varphi \vee \psi$	upper bound
(lb_1)	$\varphi \wedge \psi \to \varphi$	lower bound
(lb_2)	$\varphi \wedge \psi \to \psi$	lower bound
(sup)	$(\varphi \to \xi) \wedge (\psi \to \xi) \to (\varphi \vee \psi \to \xi)$	supremality
(inf)	$(\xi \to \varphi) \wedge (\xi \to \psi) \to (\xi \to \varphi \wedge \psi)$	infimality
(push)	$\varphi \to (\mathtt{t} \to \varphi)$	push
(pop)	$(\mathtt{t} \to \varphi) \to \varphi$	pop
(res_1)	$(\varphi \to (\psi \to \xi)) \Leftrightarrow (\varphi \,\&\, \psi \to \xi)$	residuation
(MP)	$\varphi, \varphi \to \psi \blacktriangleright \psi$	*modus ponens*
(Adj)	$\varphi, \psi \blacktriangleright \varphi \wedge \psi$	adjunction

Table: The axiomatic system of Abelian logic

Łukasiewicz Unbound logic (Łu)

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We define Łukasiewicz unbound logic Łu as $\mathrm{Ab} + (\mathtt{f} \vee \varphi \blacktriangleright \varphi)$.

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Theorem

Let $\Gamma \cup \{\varphi\}$ *be a finite set of formulas. Then, we have*

$$\Gamma \vdash_{\text{Ł}} \varphi \qquad \textit{iff} \qquad \tau[\Gamma] \vdash_{\mathrm{Lu}} \tau(\varphi).$$

Product Łukasiewicz logic (PŁ and PŁ$'$)

Product Łukasiewicz logic (PŁ and PŁ′)

Recall that Product Łukasiewicz logic PŁ is defined as the expansion of Łukasiewicz logic with the additional product operation $\odot$ satisfying the following axioms:

(Dis)	$(\chi \odot \varphi) \ominus (\chi \odot \psi) \Leftrightarrow \chi \odot (\varphi \ominus \psi)$	Distributivity
(Ass)	$\varphi \odot (\psi \odot \chi) \Leftrightarrow (\varphi \odot \psi) \odot \chi$	Associativity
(Un)	$\varphi \to \varphi \odot \overline{1}$	Unit element
(Mon)	$\varphi \odot \psi \to \varphi$	Monotonicity
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Additionally, we define PŁ$'$ as a expansion of PŁ by the rule

(ZD) $-(\varphi \cdot \varphi) \blacktriangleright \varphi$ zero divisors

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Recall that Product Łukasiewicz logic $\mathrm{PŁ}$ is defined as the expansion of Łukasiewicz logic with the additional product operation $\odot$ satisfying the following axioms:

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Theorem

The logic PŁ$'$ is strongly finitely complete w.r.t. standart PL-algebra $[0,1]_{\mathbb{R}}$.

Lattice ordered rings (ℓ-rings)

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Definition

An algebra $\boldsymbol{A} = \langle A, +, -, 0, \cdot, \wedge, \vee \rangle$ is a *lattice ordered ring* (ℓ-ring) if

1. $\langle A, +, -, 0, \cdot \rangle$ is a ring,
2. $\langle A, +, -, 0, \wedge, \vee \rangle$ is an Abelian ℓ-group,
3. $a \cdot b \geq 0$ for all $a, b \geq 0$.

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3. $a \cdot b \geq 0$ for all $a, b \geq 0$.

Moreover, we say that ℓ-ring is *commutative*, if $\cdot$ is commutative and *unital* A possed additional constant 1, which satisfies $1 \cdot a = a \cdot 1 = a$ for each a.

The logic of commutative unital ℓ-rings (Rng)

$$\mathrm{Ab}+$$

(Dis)	$\varphi \cdot (\psi \to \chi) \Leftrightarrow ((\varphi \cdot \psi) \to (\varphi \cdot \chi))$	distributivity
(Ass)	$(\varphi \cdot \psi) \cdot \chi \Leftrightarrow \varphi \cdot (\psi \cdot \chi)$	associativity
(Com)	$\varphi \cdot \psi \to \psi \cdot \varphi$	commutativity
(Neg)	$\mathtt{f} \cdot \varphi \Leftrightarrow -\varphi$	negation
(Op)	$\varphi, \psi \blacktriangleright \varphi \cdot \psi$	order-preservation
(Cng)	$\varphi, -\varphi \blacktriangleright \varphi \cdot \psi$	0-multiplication

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Let **A** *be an* ℓ*-ring. We say* **A** *is* f*-ring, if* **A** *satisfies the following quasiequation:*

$$x \wedge y = 0, z \geq 0 \implies xz \wedge y = zx \wedge y = 0.$$

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This corresponds to the following rule

$$-\varphi, -\psi, \varphi \vee \psi \blacktriangleright (\varphi \cdot \xi) \vee \psi \tag{Ort}$$

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Theorem

The logic $\mathrm{Rng} + (\mathrm{Ort})$ *is the minimal semilinear extension of* Rng*.*

Integral domains

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Lemma

Let $\mathbf{A}$ *be a totally ordered* f*-ring.* $\mathbf{A}$ *is integral domain if and only if* $\mathbf{A}$ *satisfies the following quasiequation:*

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This corresponds to the following rule

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Definition

We define Product Łukasiewicz Unbound Logic PŁu' as $\mathrm{Rng} + (\mathrm{Ort}) + (\mathrm{ZD})$

Completeness Theorem for PŁu'

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Lemma

Every ordered integral domain can be embedded into an ordered field.

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Every ordered field can be embedded into an ordered real closed field.

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Theorem

All real closed fields are elementarily equivalent.

Corollary

Let $\mathbf{A}$ *be an arbitary real closed field. The logic PŁu' is finitely strongly complete w.r.t.* $\boldsymbol{A}$.

Translation of PŁ$'$ into PŁu'

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We define a new operation $\odot$ as follows:

$$\varphi \odot \psi := -f \to (f \to \varphi) \cdot (f \to \psi)$$

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Translation of PŁ′ into PŁu'

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In $[-1, 0]_{\mathbb{R}}$ is closed under $\odot$

Theorem

Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas. Then, we have

$$\Gamma \vdash_{PŁ'} \varphi \qquad \textit{iff} \qquad \tau[\Gamma] \vdash_{PŁu'} \tau(\varphi).$$

Unbounded product Łukasiewicz logic

Ab+

(Dis)	$\varphi \cdot (\psi \to \chi) \Leftrightarrow ((\varphi \cdot \psi) \to (\varphi \cdot \chi))$	distributivity
(Ass)	$(\varphi \cdot \psi) \cdot \chi \Leftrightarrow \varphi \cdot (\psi \cdot \chi)$	associativity
(Com)	$\varphi \cdot \psi \to \psi \cdot \varphi$	commutativity
(Neg)	$\mathtt{f} \cdot \varphi \Leftrightarrow -\varphi$	negation
(OP)	$\varphi, \psi \blacktriangleright \varphi \cdot \psi$	order-preservation
(Cng)	$\varphi, -\varphi \blacktriangleright \varphi \cdot \psi$	0-multiplication
(Ort)	$-\varphi, -\psi, \varphi \vee \psi \blacktriangleright \varphi \vee (\psi \cdot \xi)$	ortogonality
(ZD)	$-(\varphi \cdot \varphi) \blacktriangleright \varphi$	zero divisors