# Spectral spaces from closure operators on semilattice-ordered semigroups

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TACL 2026

July 30, 2026

# Semilattice-ordered semigroups

## Definition (Semilattice-ordered semigroup)

A *semilattice-ordered semigroup* is triple $(S, \cdot, \vee)$, where $S$ is a nonempty set, $(S, \cdot)$ is a semigroup, $(S, \vee)$ is a semilattice and

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for all $x, y, z \in S$.

Any semilattice-ordered semigroup is a partially ordered set in which the binary relation $x \leqslant y$ is defined as follows:

$$x \leqslant y \iff x \vee y = y.$$

It is easy to see that if $x \leqslant y$ and $u \leqslant v$, then $x \vee u \leqslant y \vee v$ and $xu \leqslant yv$. For subsets $A, B \subseteq S$ we write $AB := \{ab \mid a \in A, b \in B\}$.

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**Relation to additively idempotent semirings**

If $(R, +, \cdot)$ is an additively idempotent semiring, then the addition defines a semilattice operation by $x \vee y := x + y$, and $(R, \cdot, \vee)$ is a semilattice-ordered semigroup.

# Closure operators on semilattice-ordered semigroups

## Definition (Closure operator)

Let $S$ be a semilattice-ordered semigroup. A *closure operator* on $S$ is a map $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ such that for all $A, B \in \mathcal{P}(S)$,

(1) $A \subseteq \mathrm{cl}(A)$,

(2) $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$,

(3) $A \subseteq B \implies \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

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For notational convenience, we let $\mathrm{cl}(A) := A^{\mathrm{cl}}$ for all $A \in \mathcal{P}(S)$. A subset $A$ is said to be $\mathrm{cl}$-*closed* if $A^{\mathrm{cl}} = A$.

## Definition (Algebraic closure operator)

Let $S$ be a semilattice-ordered semigroup. A closure operator $\mathrm{cl}$ on $S$ is said to be *algebraic* (or *of finite type*) if

$$A^{\mathrm{cl}} = \bigcup \left\{ F^{\mathrm{cl}} \mid F \in \mathcal{P}_{\mathsf{fin}}(A) \right\}$$

for any $A \in \mathcal{P}(S)$.

# Spectral spaces

$T_0$-axiom: For any two points $x, y$ in topological space $X$, there is an open set $U$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

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**Definition (Spectral space)**

A *spectral space* is a quasi-compact and $T_0$ topological space $X$ such that:

(1) any finite intersection of quasi-compact open subsets of $X$ is a quasi-compact open subset and the family of all quasi-compact open subsets of $X$ forms a basis of $X$,

(2) any nonempty irreducible closed subset of $X$ has a unique generic point.

**Definition (Spectral map, retrocompact subset and patch topology)**

Let $X, Y$ be a spectral spaces.

(1) A map $f : X \to Y$ is said to be *spectral* if for every quasi-compact open subset $V$ of $Y$, $f^{-1}(V)$ is quasi-compact and open.

(2) A subset $W$ of $X$ is said to be *retrocompact* in $X$ if for every quasi-compact open subset $U$ of $X$, $U \cap W$ is quasi-compact.

(3) A *patch* (or *constructible*) topology on $X$ is the coarsest topology for which all quasi-compact open subsets of $X$ are clopen. We denote by $X^{\text{cons}}$ the space $X$, with the patch topology.

# Hull-kernel topology

Given a set $S$, the power set $\mathcal{P}(S)$ is a spectral space endowed with the *hull-kernel topology* whose open subbase is a spectral space. For an arbitrary subset $E \subseteq S$, define

$$D(E) := \{A \in \mathcal{P}(S) \mid E \nsubseteq A\}, \quad V(E) := \{A \in \mathcal{P}(S) \mid E \subseteq A\}.$$

Thus, $V(E) = \mathcal{P}(S) \setminus D(E)$. The family $\{D(F)\}_{F \in \mathcal{P}_{\mathrm{fin}}(S)}$ forms an open subbasis of the hull-kernel topology on $\mathcal{P}(S)$. Moreover, each set of this family is quasi-compact in $\mathcal{P}(S)$.

# The fixed-point space

**Theorem**

*Let $S$ be a semilattice-ordered semigroup and let $\mathrm{cl}$ be closure operator on $S$. Put*

$$X := \left\{A \in \mathcal{P}(S) \mid A^{\mathrm{cl}} = A\right\}$$

*and endow it with the subspace topology induced by the hull-kernel topology on $\mathcal{P}(S)$. Then*

$$\boxed{\mathrm{cl} \text{ is algebraic}} \iff \boxed{\begin{array}{c} X \text{ is a spectral space and} \\ \text{a retrocompact subset of } \mathcal{P}(S). \end{array}}$$

# Why algebraicity is essential?

Let $S = [0,1]$ be the real unit interval, with the semigroup operations $x \cdot y = x \vee y := \min(x, y)$. Define the closure operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$A^{\mathrm{cl}} = \begin{cases} \varnothing, & A = \varnothing, \\ [0, \sup(A)], & A \neq \varnothing. \end{cases}$$

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For $A = \{1 - \frac{1}{n} \mid n \geqslant 1\}$,

$$A^{\mathrm{cl}} = [0,1], \qquad \bigcup_{F \in \mathcal{P}_{\mathrm{fin}}(A)} F^{\mathrm{cl}} = [0,1),$$

so $\mathrm{cl}$ is not algebraic.

Let $X$ be defined as in the previous theorem and let $F \neq \varnothing$ be a finite subset of $S$. Then

$$X = \{\varnothing\} \cup \{[0,x] \mid x \in [0,1]\}$$

and

$$D_X(F) := D(F) \cap X = \{\varnothing\} \cup \{[0,x] \mid 0 \leqslant x < \sup(F)\}.$$

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Hence an open subbasis of $X$ is given by the sets of the form

$$D_t := \{\varnothing\} \cup \{[0,x] \mid 0 \leqslant x < t\}$$

for $t \in [0,1]$. For all $s,t \in [0,1]$ we have $D_s \cap D_t = D_{\min(s,t)}$ and for every family $\{t_i\}_{i\in I} \subseteq [0,1]$, it holds $\bigcup_{i\in I} D_{t_i} = D_{\sup_{i\in I} t_i}$. Therefore, every proper open subset of $X$ is of the form $D_t$ for some $t \in [0,1]$.

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Let us recall that a spectral space must have a basis of quasi-compact open sets. But for every $t > 0$ the set $D_t$ is not quasi-compact. Indeed, if $t_n$ is an increasing sequence, $t_n \to t$, and $t_n < t$ for every natural number $n$, then $D_t = \bigcup_{n \in \mathbb{N}} D_{t_n}$ has no finite subcover.

# Transporting multiplication to $X$

Having constructed a spectral space from an algebraic closure operator, the next natural question is whether the semigroup structure of $S$ can be transported to that space in a topologically meaningful way. Define an operation $\star : X \times X \to X$ by

$$A \star B = (AB)^{\mathrm{cl}}.$$

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**Proposition**

*If* $\mathrm{cl}$ *is multiplicative, i.e.* $(AB)^{\mathrm{cl}} = (A^{\mathrm{cl}} B^{\mathrm{cl}})^{\mathrm{cl}}$, *then* $(X, \star)$ *is a semigroup.*

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We shall denote by $X_{\mathrm{fin}}$ the family of all finitely generated subsets of $X$, i.e. $X_{\mathrm{fin}} := \{F^{\mathrm{cl}} \mid F \in \mathcal{P}_{\mathrm{fin}}(S)\}$.

# Spectrality of the induced multiplication

**Theorem**

*Let $S$ be a semilattice-ordered semigroup and let $\mathrm{cl}$ be an algebraic closure operator on $S$. Then the following conditions are equivalent:*

(1) *the map $\star : X \times X \to X$ is spectral,*

(2) *for every finite subset $F \subseteq S$ there exists a finite (possibly empty) family of pairs $(G_i, H_i) \in \mathcal{P}_{\mathrm{fin}}(S) \times \mathcal{P}_{\mathrm{fin}}(S)$, $i \in I$, such that for all $A, B \in X$,*

$$F \subseteq A \star B \iff \exists i \in I \text{ such that } G_i \subseteq A \text{ and } H_i \subseteq B,$$

(3) *for every finite subset $F \subseteq S$ there exists a finite (possibly empty) family of pairs $(G_i, H_i) \in X_{\mathrm{fin}} \times X_{\mathrm{fin}}$, $i \in I$, such that for all $A, B \in X_{\mathrm{fin}}$,*

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# A well quasi-order criterion

**Definition (Well-quasi ordered set)**

A preordered set $(P, \leqslant)$ is *well-quasi-ordered* if every infinite sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $P$ contains an increasing pair, i.e. there exist indices $m < n$ such that $x_m \leqslant x_n$.

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**Proposition**

*Let $S$ be a semilattice-ordered semigroup and let $\mathrm{cl}$ be an algebraic closure operator on $S$. If the poset $(X_{fin}, \subseteq)$ is well-quasi-ordered, then the map $\star : X \times X \to X$ is spectral.*

# Applications: a downward-join closure

**Proposition**

*Let $S$ be a semilattice-ordered semigroup. Then the assignment*

$$A \mapsto A^{\mathrm{dj}} := \left\{x \in S \mid \exists a_1, \ldots, a_n \in A \text{ such that } x \leqslant a_1 \vee \ldots \vee a_n\right\}$$

*defines a closure operator on $S$ (called downward-join closure).*

The downward-join closure is described only in terms of the order and the join structure. In fact, $A^{\mathrm{dj}}$ is the least subset of $S$ containing $A$, which is both a down set and closed under finite joins.

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The downward-join closure is described only in terms of the order and the join structure. In fact, $A^{\mathrm{dj}}$ is the least subset of $S$ containing $A$, which is both a down set and closed under finite joins.
Since the downward-join closure is algebraic, the set

$$X_{\mathrm{dj}} := \{A \in \mathcal{P}(S) \mid A^{\mathrm{dj}} = A\},$$

endowed with the subspace topology induced by the hull-kernel topology on $\mathcal{P}(S)$, is a spectral space. Moreover, it can be shown that it is always multiplicative, so $(X_{\mathrm{dj}}, \star_{\mathrm{dj}})$ is a semigroup, where

$$\star_{\mathrm{dj}} : X_{\mathrm{dj}} \times X_{\mathrm{dj}} \to X_{\mathrm{dj}}, \quad A \star_{\mathrm{dj}} B := (AB)^{\mathrm{dj}}.$$

**Proposition**

*Let $S$ be a semilattice-ordered semigroup and let $\mathrm{cl}$ be an algebraic closure operator on $S$ satisfying the following conditions:*

(1) *for every finite nonempty subset $F \subseteq S$ there exists an element $x_F \in S$ such that $F^{\mathrm{cl}} = \{x_F\}^{\mathrm{cl}}$,*

(2) *for all $x, y \in S$,*
$$x \leqslant y \implies \{x\}^{\mathrm{cl}} \subseteq \{y\}^{\mathrm{cl}}.$$

*If the poset $(S, \leqslant)$ is well-quasi-ordered, then the poset $(X_{\mathit{fin}}, \subseteq)$ is also well-quasi-ordered.*

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## Corollary

*Let $S$ be a semilattice-ordered semigroup. If the poset $(S, \leqslant)$ is well-quasi-ordered, then the map $\star_{\mathrm{dj}} : X_{\mathrm{dj}} \times X_{\mathrm{dj}} \to X_{\mathrm{dj}}$ is spectral.*

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Thank you for your attention!