Conjunctive Concept Algebras

Named and Unnamed Perspective

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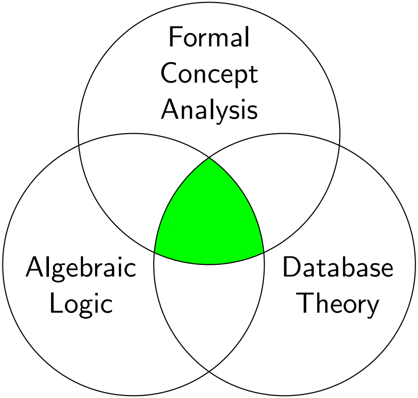
TACL 2026

Kraków, Poland, 27–31 July 2026

Overview

Overview



Formal Concept Analysis (FCA)

Formal Context

A formal context is a triple $\mathbb{K} = (G, M, I)$ consisting of

- a set G of objects
- a set M of attributes
- an incidence relation $I \subseteq G \times M$

Addams Family	male	female	black hair	psychic visions	electrokinesis
Wednesday		×	×	×	
Gomez	×		×		
Pugsley	×				
Morticia		×	×	×	
Fester	×				×

Maps on a Formal Context

Two maps are associated with each formal context.

1. Each attribute set $B \subseteq M$ defines a set of objects.

$$(\cdot)^{\downarrow} : \mathfrak{P}(M) \to \mathfrak{P}(G)$$

$$B^{\downarrow} := \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}$$

2. "What do the objects of $A \subseteq G$ have in common?"

$$(\cdot)^{\uparrow} : \mathfrak{P}(G) \to \mathfrak{P}(M)$$

$$A^{\uparrow} := \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}$$

Addams Family	male	female	black hair	psychic visions	electrokinesis
Wednesday		×	×	×	
Gomez	×		×		
Pugsley	×				
Morticia		×	×	×	
Fester	×				×

Galois Connection

The maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ form an antitone Galois connection between the power set lattices $(\mathfrak{P}(G), \subseteq)$ and $(\mathfrak{P}(M), \subseteq)$.

- $A_1 \subseteq A_2 \Rightarrow A_2^{\uparrow} \subseteq A_1^{\uparrow}$
- $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$
- $A \subseteq A^{\uparrow\downarrow}$
- $B \subseteq B^{\downarrow\uparrow}$

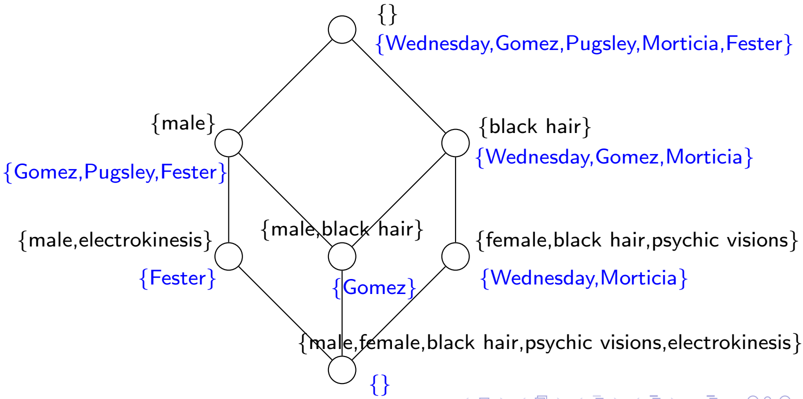
The image of each map forms a closure system.

Concept Lattice

The closure systems of an antitone Galois connection are dually isomorphic. Pairing them up results in the concept lattice of $\mathbb{K}$:

$$\mathfrak{B}(\mathbb{K}) := \{(A, B) \in \mathfrak{P}(G) \times \mathfrak{P}(M) \mid A^{\uparrow} = B \text{ and } B^{\downarrow} = A\} .$$

We call A the extent and B the intent of a concept (A, B).



Basic Theorem on Concept Lattices

Basic Theorem on Concept Lattices

The concept lattices are (u.t.i.) precisely the complete lattices.

Galois Connections between Complete Lattices

The maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ are supremum-reversing, i.e.

$$(A_1 \cup A_2)^{\uparrow} = A_1^{\uparrow} \cap A_2^{\uparrow} \,,$$
$$(B_1 \cup B_2)^{\downarrow} = B_1^{\downarrow} \cap B_2^{\downarrow} \,,$$

and likewise for arbitrary unions.

Galois Connections between Complete Lattices

A map $f : U \to V$ between complete lattices has a dual adjoint if and only if it is supremum-reversing, i.e.

$$f(\bigvee_{i \in I} u_i) = \bigwedge_{i \in I} f(u_i).$$

A Logical Perspective

- A finite attribute set $B \subseteq M$ represents a formula, e.g. $\{\mathrm{black\_hair}, \mathrm{female}\} \,\widehat{=}\, \mathrm{black\_hair}(x) \wedge \mathrm{female}(x)$.
- Each attribute is considered to be a unary relation symbol.
- Formulas are conjunctions of atoms in a single variable x.
- The structural representation of formulas by attribute sets provides an infinitary logic (only relevant for infinite $\mathbb{K}$).
- The concept extents are the definable object sets in this logic.

Motivation: FCA with Relations

Generalize the approach to primitive-positive formulas (i.e. different variables, relation symbols of any arity, using $\wedge$ and $\exists$).

FCA + Database Theory

Relational Signature

A relational signature is

- A set M of relation names
- Each $m \in M$ has an arity $|m| \in \{1, 2, 3, \dots\}$

Example (Family Relation Names)

$M_{\text{fam}} := \{\text{male}/1, \text{female}/1, \text{mother}/2, \text{father}/2, \text{parent}/2\}$

A relational structure $\mathfrak{G}$ over M (also: M-structure) consists of

- an underlying set or universe G
- for each n-ary relation name m:
  an n-ary relation $m^{\mathfrak{G}} \subseteq G^n$

Relational Structure $\mathfrak{G}_{\text{fam}}$

Three Mental Models

male
Bob
Chris

mother	
Anne	Bob
Anne	Chris

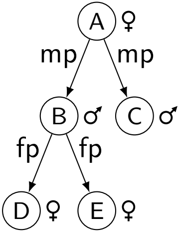
parent	
Anne	Bob
Anne	Chris
Bob	Dora
Bob	Emily

female
Anne
Dora
Emily

father	
Bob	Dora
Bob	Emily

$\mathbb{K}_1$	male	female
Anne		×
Bob	×	
Chris	×	
Dora		×
Emily		×

$\mathbb{K}_2$	parent	mother	father
(Anne,Bob)	×	×	
(Anne,Chris)	×	×	
(Bob,Dora)	×		×
(Bob,Emily)	×		×



Primitive Positive Formulas

An atomic formula or atom over M is either:

- a relational atom $m(x_1, \ldots, x_n)$
- an equality atom $x = y$

A primitive-positive formula over M is built from atoms using:

- the conjunction $\varphi \wedge \psi$
- the existential quantification $\exists x\ \varphi$

Notation

$\mathrm{PP}(M)$: The set of primitive-positive formulas over M

Conjunctive Queries

Logical Representation

A conjunctive query is a primitive-positive formula. The result table for a conjunctive query φ in an M-structure $\mathfrak{G}$ is the set

$$\mathrm{res}_{\mathfrak{G}}(\varphi) := \{t \in G^{\mathsf{free}(\varphi)} \mid \mathfrak{G} \models \varphi[t]\} \ .$$

Example

Conjunctive Query

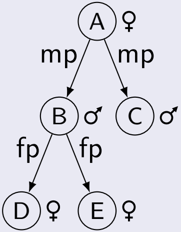
$$\exists z\ (\mathrm{mother}(x, z) \wedge \mathrm{parent}(z, y))$$

$$\downarrow \mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}$$

Result Table

x	y
Anne	Dora
Anne	Emily

Data



Tables

A table is a set $T \subseteq G^X$ where

- G is the set of permitted table entries
- $X \subseteq \mathbf{var}$ is a finite set of column names
- each row $t \in T$ is a map $t : X \to G$

The tables with entries in G are collected in the set

$$\mathrm{Tab}(G) = \bigcup_{X \subseteq_{\mathrm{fin}} \mathbf{var}} \mathfrak{P}(G^X)$$

Schema of a Table

The schema of a table $T \in \mathrm{Tab}(G)$ is

$$\mathrm{schema}(T) := \begin{cases} X & \text{if } T \in G^X \text{ and } T \neq \emptyset \\ \textbf{var} & \text{if } T = \emptyset \end{cases}$$

Natural Join

The result operation $\mathrm{res}_{\mathfrak{G}} : \mathrm{PP}(M) \to \mathrm{Tab}(G)$ translates conjunction of formulas to the natural join of tables:

$$\mathrm{res}_{\mathfrak{G}}(\varphi \wedge \psi) = \mathrm{res}_{\mathfrak{G}}(\varphi) \bowtie \mathrm{res}_{\mathfrak{G}}(\psi)$$

Example

mother(x, z) $\xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

x	z
Anne	Bob
Anne	Chris

$\bowtie$

parent(z, y) $\xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

z	y
Anne	Bob
Anne	Chris
Bob	Dora
Bob	Emily

$=$

mother$(x, z) \wedge$ parent(z, y) $\xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

x	z	y
Anne	Bob	Dora
Anne	Bob	Emily

Natural Join

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$$\mathrm{res}_{\mathfrak{G}}(\varphi \wedge \psi) = \mathrm{res}_{\mathfrak{G}}(\varphi) \bowtie \mathrm{res}_{\mathfrak{G}}(\psi)$$

Example

$\mathrm{mother}(x, z) \xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

x	z
Anne	Bob
Anne	Chris

$\bowtie$

$\mathrm{parent}(z, y) \xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

z	y
Anne	Bob
Anne	Chris
Bob	Dora
Bob	Emily

$=$

$\mathrm{mother}(x, z) \wedge \mathrm{parent}(z, y) \xrightarrow{\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}}$

x	z	y
Anne	Bob	Dora
Anne	Bob	Emily

The natural join of $S \subseteq G^X$ and $T \subseteq G^Y$ is the table

$$S \bowtie T := \{t \in G^{X \cup Y} \mid t|_X \in S \text{ and } t_Y \in T\}$$

Motivation: Table Order

Does the result operation have an adjoint ?

$$\mathrm{res}_{\mathfrak{G}}(\varphi \wedge \psi) = \mathrm{res}_{\mathfrak{G}}(\varphi) \bowtie \mathrm{res}_{\mathfrak{G}}(\psi)$$

Tropashko Lattice

- The natural join is associative, commutative and idempotent
- Hence $(\mathrm{Tab}(G), \bowtie)$ is a meet-semilattice
- The implicit table order is given by $T_1 \leq T_2 :\Leftrightarrow T_1 = T_1 \bowtie T_2$
- The poset $(\mathrm{Tab}(G), \leq)$ is a complete lattice, essentially the Tropashko lattice [LMH16][ST06]

Inner Union / Co-Join

The inner union $S \oplus T$ (or co-join $S \bowtie T$) of tables is the supremum operation in the Tropashko lattice.

Example

x	z
Anne	Bob
Anne	Chris

$\oplus$

z	y
Bob	Dora
Bob	Emily

$=$

z
Bob
Chris

Inner Union / Co-Join

The inner union $S \oplus T$ (or co-join $S \bowtie T$) of tables is the supremum operation in the Tropashko lattice.

Example

x	z
Anne	Bob
Anne	Chris

$\oplus$

z	y
Bob	Dora
Bob	Emily

$=$

z
Bob
Chris

The inner union of table $S \subseteq G^X$ and $T \subseteq G^Y$ is the table

$$S \oplus T := \{s|_{X \cap Y} \mid s \in S\} \cup \{t|_{X \cap Y} \mid t \in T\} .$$

Table Order

Table Order (“Project and Compare”)

Let $T_1, T_2 \in \mathrm{Tab}(G)$. Then

$$T_1 \leq T_2 :\Leftrightarrow \begin{cases} \quad \mathsf{schema}(T_1) \supseteq \mathsf{schema}(T_2) \\ \text{and } \{t|_{\mathsf{schema}(T_2)} \mid t \in T_1\} \subseteq T_2 \end{cases} .$$

Example

x	y	z
Anne	Bob	Chris
Bob	Chris	Dora

$\leq$

y	z
Bob	Chris
Chris	Dora
Anne	Emily

Motivation: Query Order

If $\mathrm{res}_{\mathfrak{G}}$ has an adjoint, then it has to be monotone:

$$\varphi \models \psi \;\Rightarrow\; \mathrm{res}_{\mathfrak{G}}(\varphi) \leq \mathrm{res}_{\mathfrak{G}}(\psi)$$

But this requires $\mathsf{free}(\varphi) \supseteq \mathsf{free}(\psi)$.

Table Semantics

- A named tuple over G is a map $t : X \to G$, defined on a finite set of variables
- A variable $x \in \mathbf{var}$ is undefined under t if $x \notin \mathrm{def}(t)$

Notation

$\mathrm{NTup}(G)$: The set of named tuples over G

Table Semantics

A formula $\varphi \in \mathrm{PP}(M)$ holds in an M-structure G under a variable assignment $t \in \mathrm{NTup}(G)$, written $(\mathfrak{G}, t) \models_{\mathrm{tbl}} \varphi$, if and only if

- t is defined on all free variables of φ
- φ holds in $\mathfrak{G}$ under t in the usual sense of standard logic

Standard Semantics vs Table Semantics

Logical Implication

$\varphi \models_{\mathrm{tbl}} \psi \;\Leftrightarrow\; \varphi \models \psi$ and free$(\varphi) \supseteq$ free(ψ)

Logical Equivalence

$\varphi \mathrel{=\!\!|\!\models_{\mathrm{tbl}}} \psi \;\Leftrightarrow\; \varphi \mathrel{=\!\!|\!\models} \psi$ and free$(\varphi) =$ free(ψ)

A Note on Table Semantics

The formulas $x{=}x$, $y{=}y$ and $x{=}x \wedge y{=}y$ are equivalent under standard semantics (being tautologies), but not as queries

$\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}(x{=}x)$:

x
Anne
Bob
Chris
Dora
Emily

$\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}(y{=}y)$:

y
Anne
Bob
Chris
Dora
Emily

$\mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}(x{=}x \wedge y{=}y)$:

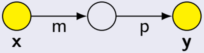
x	y
Anne	Anne
Anne	Bob
Anne	Chris
Anne	Dora
Anne	Emily
Bob	Anne
Bob	Bob
Bob	Chris
Bob	Dora
Bob	Emily
Chris	Anne
Chris	Bob
Chris	Chris
Chris	Dora
Chris	Emily
Dora	Anne
Dora	Bob
Dora	Chris
Dora	Dora
Dora	Emily
Emily	Anne
Emily	Bob
Emily	Chris
Emily	Dora
Emily	Emily

Tableau Queries

A tableau query over M is a pair $(\mathfrak{A}, \lambda)$ consisting of

- an M-structure $\mathfrak{A}$
- a window $\lambda : X \to A$ where $X \subseteq_{\text{fin}}$ **var**

Examples

Logical Query	Tableau Query
$\exists z\ (\text{mother}(x, z) \wedge \text{parent}(z, y))$	
$x = y$	
$x = x$	
$\exists x\ x = x$	

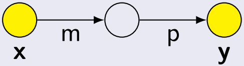
Result Operation for Tableau Queries

The following result operation is equivalent to the previous result operation for logical queries:

$$\mathrm{res}_{\mathfrak{G}}(\mathfrak{A}, \lambda) := \{ f \circ \lambda \mid f : \mathfrak{A} \to \mathfrak{G} \}$$

Example

Graph Query

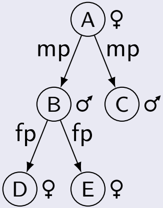


$\Big\downarrow \mathrm{res}_{\mathfrak{G}_{\mathrm{fam}}}$

Result Table

x	y
Anne	Dora
Anne	Emily

Data



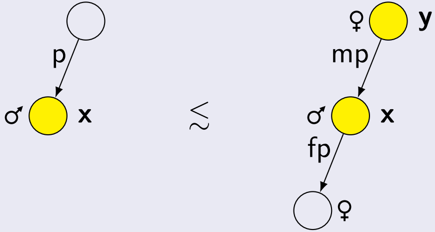
Homomorphisms of Tableau Queries

A homomorphism $f : (\mathfrak{A}, \lambda) \to (\mathfrak{B}, \mu)$ of tableau queries is a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ with $f \circ \lambda \subseteq \mu$

Homomorphism Preorder

$(\mathfrak{A}, \lambda) \lesssim (\mathfrak{B}, \mu)$:⇔ There exists $f : (\mathfrak{A}, \lambda) \to (\mathfrak{B}, \mu)$

Example



The Meaning of Homomorphisms

Homomorphism Theorem

Let φ, ψ be formulas in $\mathrm{PP}(M)$. Let $\Gamma(\varphi), \Gamma(\psi)$ be their representations as tableau queries. Then

$$\varphi \models_{\mathrm{tbl}} \psi \;\Leftrightarrow\; \Gamma(\psi) \lesssim \Gamma(\varphi)\ .$$

Category of Tableau Queries

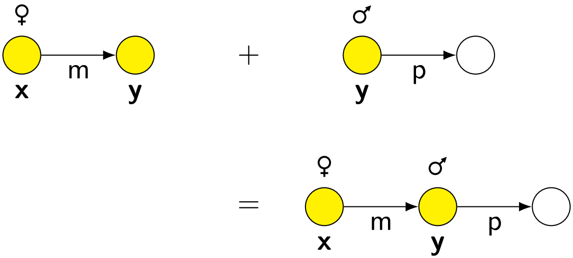
The class $\mathrm{TQ}(M)$ of tableau queries, together with their homomorphisms, forms the category of tableau queries.

The category has arbitrary products and coproducts, which are infima and suprema in the homomorphism preorder.

$$
\begin{tikzcd}
 & \mathfrak{S} \arrow[d, "f"] \arrow[ddl, bend right, "f_j"'] \arrow[ddr, bend left, "f_k"] & \\
 & \prod_{i \in I} \mathfrak{T}_i \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\
\mathfrak{T}_j & & \mathfrak{T}_k
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{T}_j \arrow[dr, "\iota_j"] \arrow[ddr, bend right, "f_j"'] & & \mathfrak{T}_k \arrow[dl, "\iota_k"'] \arrow[ddl, bend left, "f_k"] \\
 & \sum_{i \in I} \mathfrak{T}_i \arrow[d, "f"] & \\
 & \mathfrak{S} &
\end{tikzcd}
$$

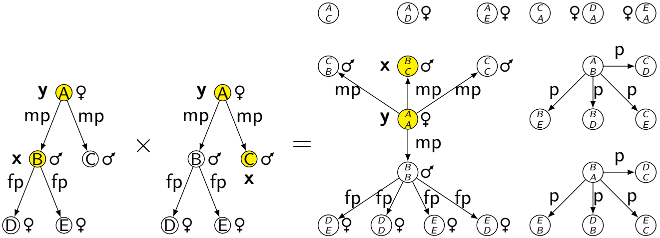
Sum of Tableau Queries

The sum of tableau queries represents the conjunction of formulas. The graphs are merged at yellow nodes that represent the same free variable.



Product of Tableau Queries

The product of tableau queries provides an abstraction mechanism.



Galois Connection

The result operation $\mathrm{res}_{\mathfrak{G}} : \mathrm{TQ}(M) \to \mathrm{Tab}(G)$ has a dual adjoint $\mathrm{info}_{\mathfrak{G}} : \mathrm{Tab}(G) \to \mathrm{TQ}(M)$.

The dual adjoint shows what the tuples in a table have in common:

$$\mathrm{info}_{\mathfrak{G}}(T) := \bigtimes_{t \in T} (\mathfrak{G}, t)$$

The Link between Table Operations and Query Operations

$$\mathrm{res}_{\mathfrak{G}}(Q_1 + Q_2) = \mathrm{res}_{\mathfrak{G}}(Q_1) \bowtie \mathrm{res}_{\mathfrak{G}}(Q_2)$$
$$\mathrm{info}_{\mathfrak{G}}(T_1 \oplus T_2) = \mathrm{info}_{\mathfrak{G}}(T_1) \times \mathrm{info}_{\mathfrak{G}}(T_2)$$

Concept Lattice of a Relational Structure

Concept Lattice

$$\mathfrak{B}(\mathfrak{G}) := \{(T, Q) \in \mathrm{Tab}(G) \times \mathrm{TQ}(M) \mid \mathrm{info}_{\mathfrak{G}}(T) = Q \text{ and } \mathrm{res}_{\mathfrak{G}}(Q) = T\}$$

Subconcept Order

$$(T_1, Q_1) \leq (T_2, Q_2) :\Leftrightarrow T_1 \leq T_2 \Leftrightarrow Q_2 \lesssim Q_1$$

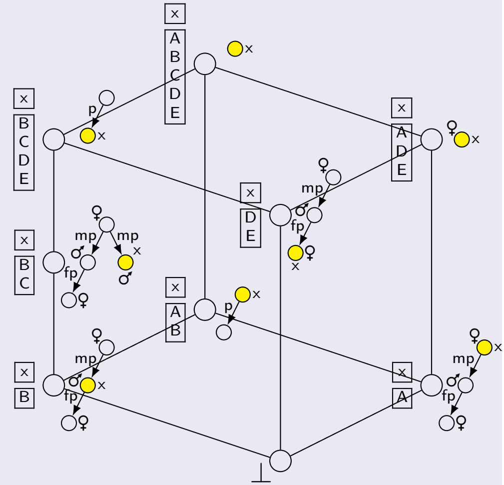
Domain

$$\mathrm{dom}(T_1, Q_1) := \mathrm{schema}(T_1) = \mathrm{free}(Q_1) \text{ , where } \mathrm{free}(\mathfrak{A}, \lambda) = \mathrm{def}(\lambda)$$

Slices

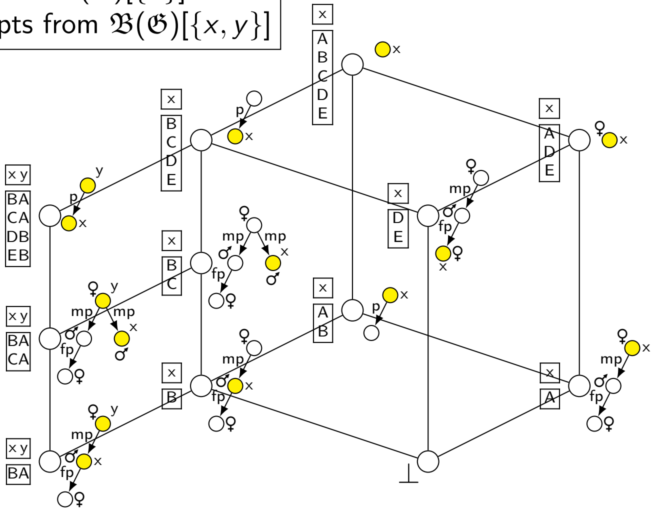
The slice $\mathfrak{B}(\mathfrak{G})[X]$ is the complete lattice of all concepts with domain X (plus $\perp$)

Example: The slice $\mathfrak{B}(\mathfrak{G}_{\text{fam}})[\{x\}]$



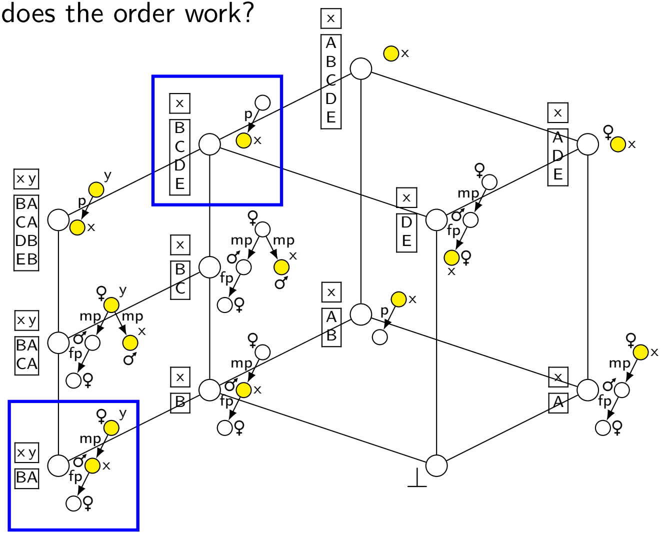
Concept Order

The slice $\mathfrak{B}(\mathfrak{G})[\{x\}]$ and
3 concepts from $\mathfrak{B}(\mathfrak{G})[\{x, y\}]$



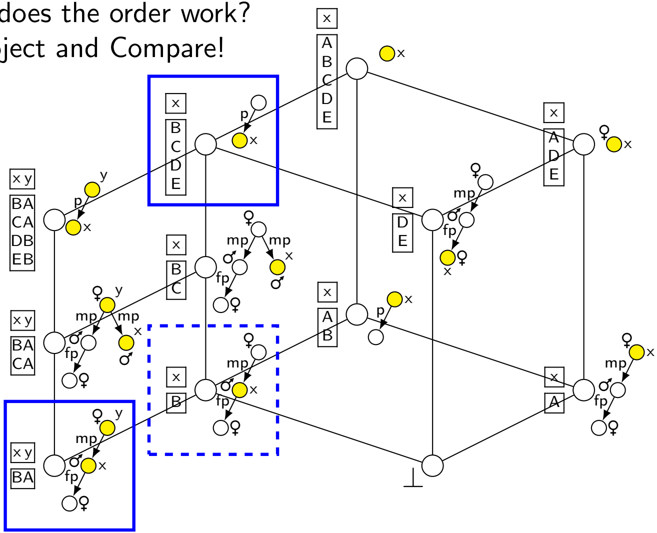
Concept Order

How does the order work?



Concept Order

How does the order work?
Project and Compare!



Conjunctive Concept Algebras – Named Perspective

Cylindrifications

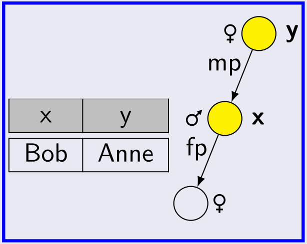
The cylindrifications $C_z : \mathfrak{B}(\mathfrak{G}) \to \mathfrak{B}(\mathfrak{G})$, $z \in \textbf{var}$, are given by

$$C_z(T, [(\mathfrak{A}, \lambda)]) := (\{t|_{\mathrm{def}(t)\setminus\{z\}} \mid t \in T\}, [(\mathfrak{A}, \lambda|_{\mathrm{def}(\lambda)\setminus\{z\}})])$$

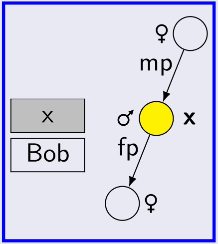
for $(T, [(\mathfrak{A}, \lambda)]) \neq \bot$. Moreover, $c_z(\bot) = \bot$.

Example

(T, Q):



$C_y(T, Q)$:



Diagonals

Let $x, y \in \mathbf{var}$. The diagonal $\mathfrak{E}_{xy} \in \mathfrak{B}(\mathfrak{G})$ is the concept

$$\mathfrak{E}_{xy} := (E_{xy}, \mathrm{info}_{\mathfrak{G}}(E_{xy})) \,,$$

where $E_{xy} := \{t \in G^{\{x,y\}} \mid t(x) = t(y)\}$.

Example

$\mathfrak{E}_{xy}$:

x	y
Anne	Anne
Bob	Bob
Chris	Chris
Dora	Dora
Emily	Emily

xy

$\mathfrak{E}_{xx}$:

x
Anne
Bob
Chris
Dora
Emily

x

Conjunctive Concept Algebras

- The full conjunctive concept algebra with base $\mathfrak{G}$ is the algebra $\underline{\mathfrak{C}}(\mathfrak{G}) := (\mathfrak{B}(\mathfrak{G}), \wedge, \bot, \top, C_x, \mathfrak{E}_{xy}, \mathrm{dom})_{x,y \in \mathbf{var}}$
- A conjunctive concept algebra with base $\mathfrak{G}$ is a subalgebra of $\underline{\mathfrak{C}}(\mathfrak{G})$

Projectional Semilattices

A projectional semilattice is an algebra $(V, \wedge, 0, 1, c_x, d_{xy}, \mathrm{dom})_{x,y \in \mathbf{var}}$ consisting of

- a bounded semilattice $(V, \wedge, 0, 1)$
- a cylindrification $c_x : V \to V$ for all $x \in \mathbf{var}$
- a diagonal $d_{xy} \in V$ for all $x, y \in \mathbf{var}$
- a domain function $\mathrm{dom} : V \to \mathfrak{P}_{\mathrm{fin}}(\mathbf{var}) \cup \{\mathbf{var}\}$

Projectional Semilattice Axioms

(PS0) $(V, \wedge, 0, 1)$ bounded semilattice

(PS1) $c_x(0) = 0$

(PS2) $u \leq c_x(u)$

(PS3) $c_x(u \wedge c_x(v)) = c_x(u) \wedge c_x(v)$

(PS4) $c_x(c_y(u)) = c_y(c_x(u))$

(PS5) $u \neq 0 \Rightarrow (u \neq c_x(u) \Leftrightarrow u \leq d_{xx})$

(PS6) $x \neq y, z \Rightarrow d_{yz} = c_x(d_{yx} \wedge d_{xz})$

(PS7) $x \neq y \Rightarrow d_{xy} \wedge c_x(d_{xy} \wedge u) \leq u$

(PS8) $u \neq 0 \Rightarrow \mathrm{dom}(u)$ finite

(PS9) $\mathrm{dom}(u) = \{x \in \mathbf{var} \mid u \leq d_{xx}\}$

(PS10) $\mathrm{dom}(u) = \emptyset \Rightarrow u = 1$

(PS11) $d_{xx} \neq 0$

(PS12) $d_{xy} = d_{yx}$

Projectional Semilattices vs Cylindric Algebras

Projectional Semilattice $(V, \wedge, 0, 1, c_x, d_{xy}, \mathrm{dom})_{x,y<\omega}$

(PS0) $(V, \wedge, 0, 1)$ is a bounded semilattice

(PS1) $c_x(0) = 0$

(PS2) $u \leq c_x(u)$

(PS3) $c_x(u \wedge c_x(v)) = c_x(u) \wedge c_x(v)$

(PS4) $c_x(c_y(u)) = c_y(c_x(u))$

(PS5) $u \neq 0 \Rightarrow (u \leq d_{xx} \Leftrightarrow u \neq c_x(u))$

(PS6) $x \neq y, z \Rightarrow d_{yz} = c_x(d_{yx} \wedge d_{xz})$

(PS7) $x \neq y \Rightarrow d_{xy} \wedge c_x(d_{xy} \wedge u) \leq u$

(PS8) $u \neq 0 \Rightarrow \mathrm{dom}(u)$ is finite

(PS9) $\mathrm{dom}(u) = \{x \in \mathbf{var} \mid u \leq d_{xx}\}$

(PS10) $\mathrm{dom}(u) = \emptyset \Rightarrow u = 1$

(PS11) $d_{xx} \neq 0$

(PS12) $d_{xy} = d_{yx}$

Cylindric Algebra $(V, \vee, \wedge, \neg, 0, 1, c_x, d_{xy})_{x,y<\omega}$

(C0) $(V, \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra

(C1) $c_x(0) = 0$

(C2) $u \leq c_x(u)$

(C3) $c_x(u \wedge c_x(v)) = c_x(u) \wedge c_x(v)$

(C4) $c_x(c_y(u)) = c_y(c_x(u))$

(C5) $d_{xx} = 1$

(C6) $x \neq y, z \Rightarrow d_{yz} = c_x(d_{yx} \wedge d_{xz})$

(C7) $x \neq y \Rightarrow c_x(d_{xy} \wedge u) \wedge c_x(d_{xy} \wedge \neg u) = 0$

(C8) $\dim(u) := \{x \in \omega \mid u \neq c_x(u)\}$ is finite

Representation Theorems

Theorem ("$\wedge$-sublattices of $\mathfrak{B}(\mathfrak{G})$")

The conjunctive concept algebras with non-empty base are (u.t.i.) precisely the projectional semilattices.

Theorem ("$\bigwedge$-sublattices of $\mathfrak{B}(\mathfrak{G})$")

The complete conjunctive concept algebras with non-empty base are (u.t.i.) precisely the complete projectional semilattices.

Conjecture ("concept lattices $\mathfrak{B}(\mathfrak{G})$")

The full conjunctive concept algebras with non-empty base are (u.t.i.) precisely the complete projectional semilattices.

Conjunctive Concept Algebras – Unnamed Perspective

Named Perspective vs Unnamed Perspective

Database Theory: Two perspectives on relations on a set G

1. Unnamed Perspective
   - a relation has an arity $n \in \omega$
   - it is a set of ordered tuples $(g_0, \ldots, g_{n-1}) \in G^n$
2. Named Perspective
   - a relation has a finite schema $I \subseteq \mathbf{att}$
   - it is a set of named tuples $t : I \to G$

Relations as Tables (Example: Airport Departures)

Unnamed Perspective

0	1	2
LH104	Munich	12:15
LH1204	Basel	12:20
LH284	Bologna	12:30

Named Perspective

Flight	Destination	Departure
LH104	Munich	12:15
LH1204	Basel	12:20
LH284	Bologna	12:30

Column Deletion

A deletion operation $c_i : \mathrm{Rel}(G) \to \mathrm{Rel}(G)$ is defined by

$$c_i(R) := \begin{cases} \{t \circ \epsilon_i^n \mid t \in R\} & \text{if } i < \#R = n \in \omega \\ R & \text{otherwise} \end{cases}$$

for each $i \in \omega$, where $\epsilon_i : n \to n-1$ is the map

$$\epsilon_i^n(\ell) := \begin{cases} \ell & \text{if } i < \ell \\ \ell + 1 & \text{otherwise} \end{cases}$$

Example: Column Deletion

R:

0	1	2
a	b	c
a	b	d
d	e	f

$c_0(R)$:

0	1
b	c
b	d
e	f

$c_1(R)$:

0	1
a	c
a	d
d	f

$c_2(R)$:

0	1
a	b
d	e

Column Insertion

An insertion operation $\eta_i : \mathrm{Rel}(G) \to \mathrm{Rel}(G)$ is defined by

$$\eta_i(R) := \begin{cases} \{t \in G^{n+1} \mid t \circ \epsilon_i^{n+1} \in R\} & \text{if } i \leq n = \#R \in \omega \\ R & \text{otherwise} \end{cases}$$

for each $i \in \omega$.

Example: Column Insertion ($G = \{a, b, c\}$)

R:

0
a

$\eta_0(R)$:

0	1
a	a
b	a
c	a

$\eta_1(R)$:

0	1
a	a
a	b
a	c

Diagonals

For all $i, j \in \omega$, the diagonal d_{ij} is the relation

$$d_{ij} = \{t \in G^{\max(i,j)+1} \mid t(i) = t(j)\}$$

Example: Column Insertion ($G = \{a, b\}$)

d_{00}:

0
a
b

d_{01}:

0	1
a	a
b	b

d_{11}:

0	1
a	a
a	b
b	a
b	b

d_{02}:

0	1	2
a	a	a
a	b	a
b	a	b
b	b	b

Projectional Skeleton

A projectional skeleton is an algebra $(U, \wedge, 0, 1, c_i, \eta_i, d_{ij}, \#)_{i,j\in\omega}$ consisting of

- a bounded semilattice $(U, \wedge, 0, 1)$
- a deletion $c_i : U \to U$ for all $i \in \omega$
- an insertion $\eta_i : U \to U$ for all $i \in \omega$
- a diagonal $d_{ij} \in U$ for all $i, j \in \omega$
- an arity function $\# : U \to \omega \cup \{\omega\}$

(PSK0)	$(U, \wedge, 0, 1)$ is a bounded semilattice
(PSK1)	$u \leq v \Rightarrow c_i u \leq c_i v$
(PSK2)	$u \leq v \Rightarrow \eta_i u \leq \eta_i v$
(PSK3)	$j \leq i \Rightarrow c_i c_j u = c_j c_{i+1} u$
(PSK4)	$j \leq i \Rightarrow \eta_j \eta_i u = \eta_{i+1} \eta_j u$
(PSK5)	$j < i \Rightarrow c_j \eta_i u = \eta_{i-1} c_j u$
(PSK6)	$j > i \Rightarrow c_j \eta_i u = \eta_i c_{j-1} u$
(PSK7)	$i \leq j, k \Rightarrow \eta_i d_{jk} = d_{j+1,k+1}$
(PSK8)	$j < i \leq k \Rightarrow \eta_i d_{jk} = d_{j,k+1}$
(PSK9)	$c_i(u \wedge \eta_i v) = c_i u \wedge v$
(PSK10)	$i \neq j \Rightarrow \eta_i c_i(u \wedge d_{ij}) \wedge d_{ij} \leq u$
(PSK11)	$c_0 d_{01} = d_{00}$
(PSK12)	$u \leq d_{i,i+1} \Rightarrow c_i u = c_{i+1} u$
(PSK13)	$d_{ij} = d_{ji}$
(PSK14)	$d_{ij} \wedge d_{jk} \leq d_{ik}$
(PSK15)	$i < \#u \Rightarrow u \leq \eta_i c_i u$
(PSK16)	$i = \#u \Rightarrow \eta_i u = u \wedge d_{ii}$
(PSK17)	$i \geq \#u \Rightarrow c_i u = u$
(PSK18)	$u \neq 0 \Rightarrow \#u$ finite
(PSK19)	$\#u = \inf\{i \in \omega \mid u \not\leq d_{ii}\}$
(PSK20)	$\#u = 0 \Leftrightarrow u = 1$
(PSK21)	$d_{i+1,i+1} < d_{ii}$

Representation Theorems

Theorem ("$\wedge$-sublattices of $\mathfrak{B}(\mathfrak{G})$")

The conjunctive concept algebras with non-empty base are (u.t.i.) precisely the projectional skeletons.

Theorem ("$\bigwedge$-sublattices of $\mathfrak{B}(\mathfrak{G})$")

The complete conjunctive concept algebras with non-empty base are (u.t.i.) precisely the complete projectional skeletons.

Conjecture ("concept lattices $\mathfrak{B}(\mathfrak{G})$")

The full conjunctive concept algebras with non-empty base are (u.t.i.) precisely the complete projectional skeletons.

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