Refutation calculi for lattice-based logics: from display to tableaux

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  Basic logic of **nondistributive** lattices with normal modal operators $+$ Belnap's display calculi.

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- **Setting.**
  Basic logic of **nondistributive** lattices with normal modal operators $+$ Belnap's display calculi.
- **Main result.**
  We introduce *refutation display calculi* whose objects are *antisequents* $\Pi \dashv \Sigma$, witnessing that $\Pi \vdash \Sigma$ is invalid.
  - **Soundness** & **Completeness** (yielding *decidability* of the LE-logic) $+$ **tableau exctraction.**

$$\frac{B \dashv C \vee D \quad A \dashv C \vee D \quad A \wedge B \dashv D \quad A \wedge B \dashv C}{A \wedge B \dashv C \vee D}$$

The refutational side of the Force

Proof theory as usually taught:

- A proof system certifies *valid* judgements.

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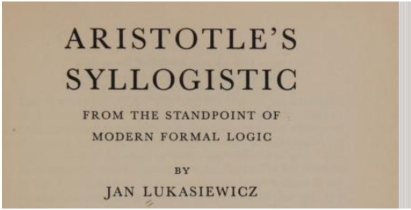
Refutation calculi

Derive *non-valid* objects positively.

$$\frac{\dashv B_1 \quad \cdots \quad \dashv B_n}{\dashv C}$$

A derivation of $\dashv A$ is a *witness* that $\nvdash A$ (not just the absence of a proof).

Refutational systems: from the origins



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SYLLOGISTIC
FROM THE STANDPOINT OF
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Refutation systems: an overview and some applications to philosophical logics

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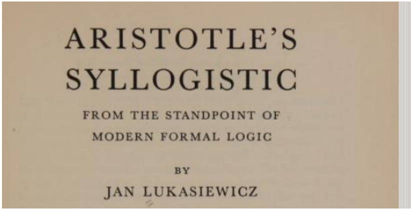
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Our approach is the first employing display calculi on lattice logic

Background: LE-logics

- LE-logics are the logics of (varieties of) **lattice expansions**: bounded lattices $+$ '$\Box$-like' and '$\Diamond$-like' normal operators.

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$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\varphi_1, \ldots, \varphi_n) \mid g(\varphi_1, \ldots, \varphi_n)$$

$f \in \mathcal{F}$, $g \in \mathcal{G}$ for arbitrary but fixed $\mathcal{F}$, $\mathcal{G}$

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$$f \in \mathcal{F},\ g \in \mathcal{G} \text{ for arbitrary but fixed } \mathcal{F}, \mathcal{G}$$

$$f(p_1 \ldots, q \vee r, \ldots p_n) = f(p_1 \ldots, q, \ldots p_n) \vee f(p_1 \ldots, r, \ldots p_n)$$

$$g(p_1 \ldots, q \wedge r, \ldots p_n) = g(p_1 \ldots, q, \ldots p_n) \wedge g(p_1 \ldots, r, \ldots p_n)$$

$$f(p_1 \ldots, \bot, \ldots p_n) = \bot$$

$$g(p_1 \ldots, \top, \ldots p_n) = \top$$

Background: LE-logics

Logic	$\mathcal{F}$-oper.	$\mathcal{G}$-oper.
Classical modal logic	$\wedge\ \Diamond\ \neg$	$\vee\ \rightarrow\ \neg\ \Box$
Intuitionistic logic	$\wedge$	$\vee\ \rightarrow$
Bi-intuitionistic logic	$\wedge\ \succ\!\!-$	$\vee\ \rightarrow$
Distributive modal logic	$\wedge\ \Diamond\ \lhd$	$\vee\ \Box\ \rhd$
Positive modal logic	$\wedge\ \Diamond$	$\vee\ \Box$
Bi-intuitionistic ML	$\wedge\ \succ\!\!-\ \Diamond$	$\vee\ \rightarrow\ \Box$
Intuitionistic modal logic	$\wedge\ \Diamond$	$\vee\ \rightarrow\ \Box$
Semi-De Morgan logic	$\wedge$	$\vee\ \neg$
Orthologic		$\neg$
Full Lambek calculus	$\otimes\ 1$	$\backslash\ /$
Lambek-Grishin calculus	$\otimes\ \backslash_\star\ /_\star\ 1$	$\star\ \backslash_\otimes\ /_\otimes\ ⅋$
Mult.-Add. linear logic	$\otimes\ 1$	$\backslash_\otimes\ ⅋$
Fundamental modal logic	$\Diamond$	$\Box\ \neg$

Background: display calculi D.LE

A formal framework that can represent different logics

$$\wedge_L \; \frac{A_i \vdash X}{A_1 \wedge A_2 \vdash X} \qquad \frac{X \vdash A_1 \qquad X \vdash A_2}{X \vdash A_1 \wedge A_2} \; \wedge_R$$

$$\Box_L \; \frac{A \vdash X}{\Box A \vdash \check{\Box} X} \qquad \frac{X \vdash \check{\Box} A}{X \vdash \Box A} \; \Box_R$$

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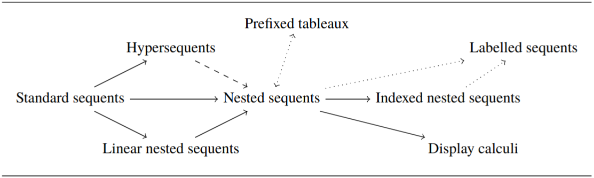
Display property: every substructure can be isolated in precedent or succedent position, via *display postulates*:

$$\frac{X \vdash \check{\Box} Y}{\hat{\blacklozenge} X \vdash Y}$$

Why the Display calculus?



THEOREM 4.4 (Elimination theorem). With respect to **DL**, or any calculus which like **DL** admits an analysis satisfying C2–C8, the "Elimination Rule" (ER) is admissible.

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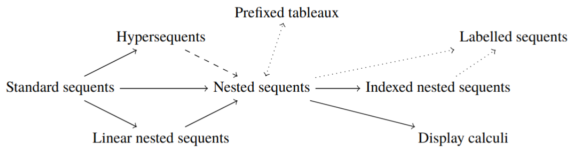

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High expressivity + cut elimination metatheorem

The refutation display calculus $\mathrm{D.LE}^r$

Axiomatic rules:

$$\frac{}{\top \dashv \bot} \qquad \frac{}{p \dashv \bot} \qquad \frac{}{\top \dashv \hat{q}} \qquad \frac{}{p \dashv q}$$

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Whitman's theorem (1941)

In any lattice, if $a \wedge b \leq c \vee d$ is valid, then at least one of

$$b \leq c \vee d, \quad a \leq c \vee d, \quad a \wedge b \leq d, \quad a \wedge b \leq c$$

is also valid.

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$$\frac{B \dashv C \vee D \quad A \dashv C \vee D \quad A \wedge B \dashv D \quad A \wedge B \dashv C}{A \wedge B \dashv C \vee D}$$

$$\vee_{L_i} \frac{A_i \dashv \Sigma}{A_1 \vee A_2 \dashv \Sigma} \qquad \frac{\Pi \dashv A_i}{\Pi \dashv A_1 \wedge A_2} \wedge_{R_i}$$

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Rules for modal connectives (no residuals):

$$\frac{(\Upsilon_i \dashv \bot \mid 1 \leq i \leq n_f) \qquad \Upsilon_j \dashv \varphi_j}{\hat{f}\,(\Upsilon_1, \ldots, \Upsilon_{n_f}) \dashv f(\varphi_1, \ldots, \varphi_{n_f})}\ f_R$$

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$$\frac{X \dashv \perp \qquad X \dashv A}{\hat{\Diamond} X \dashv \Diamond A}\ \Diamond_R$$

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General Whitman rules (Π', Σ' non-branching, no residuals):

$$\frac{\varphi \dashv \Sigma' \quad \psi \dashv \Sigma' \quad (\varphi \wedge \psi \dashv \Sigma'[A_i]^{\mathrm{pre}})_i \quad (\varphi \wedge \psi \dashv \Sigma'[B_i]^{\mathrm{pre}})_i \quad \cdots}{\varphi \wedge \psi \dashv \Sigma'[A_i \wedge B_i]^{\mathrm{pre}}[C_j \vee D_j]^{\mathrm{suc}}}\ (\wedge_L)$$

Example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax_3}
\UnaryInfC{$\hat{\top} \dashv p$}
\AxiomC{}
\RightLabel{Ax_4}
\UnaryInfC{$q \dashv p$}
\LeftLabel{$\vee_{L_2}$}
\UnaryInfC{$p \vee q \dashv p$}
\LeftLabel{$\Box_L$}
\BinaryInfC{$\Box(p \vee q) \dashv \check{\Box} p$}
\RightLabel{$\Box_R$}
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\qquad
\dfrac{
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Soundness

Two strategies to prove the soundness of

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- Syntactical approach: contrapositively, assume $\hat{\Diamond} X \vdash \Diamond A$ is derivable and prove that one of $X \vdash \bot$ and $X \vdash A$ is derivable.

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Classical picture

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- A *complete refutation calculus* for **L** gives a semi-decision procedure for *invalidity*.

- Both together $\Rightarrow$ **decidability**.
- $\mathrm{D.LE}$ (the ‘positive’ calculus) is complete.
- $\mathrm{D.LE}^r$ is complete: the difficulty in proving its completeness is analogous to finding a decision procedure for the logic.

Completeness of $\mathrm{D.LE}^r$: a decidability proof

Theorem

If $\Pi \vdash \Sigma$ is *not* derivable in $\mathrm{D.LE}$ and contains no residuals, then $\Pi \dashv \Sigma$ is derivable in $\mathrm{D.LE}^r$.

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Remark

The completeness proof implicitly uses a proof-search strategy analogous to *focused proof search* (invertible & focused phases).

Modularity in the refutational setting

The asymmetry

In $\mathrm{D.LE}$, adding a rule *extends* the logic.
In $\mathrm{D.LE}^r$, adding a rule *contracts* the logic (more invalidity derivable $\Rightarrow$ fewer valid sequents).

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Consequence: to capture an *axiomatic extension* of the basic LE logic via a refutation calculus, one must *weaken* or *restrict* rules of $\mathrm{D.LE}^r$, not add new ones.

Case studies: a work in progress

1. **Orthologic.**
2. **MALL.**
3. **Lambek calculus.**
4. **Fundamental modal logic.**

From refutation display calculi to terminating tableaux

The passage is 'automatic': take the *contrapositive* of each refutational rule.

$$\frac{\Pi \dashv A_i}{\Pi \dashv A_1 \wedge A_2} \quad \rightsquigarrow \quad \begin{array}{c} \Pi \vdash A_1 \wedge A_2 \\ | \\ \Pi \vdash A_1 \\ | \\ \Pi \vdash A_2 \end{array}$$

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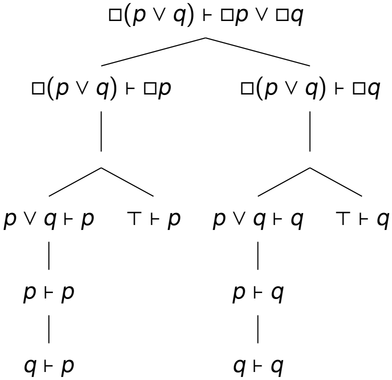
- A branch is **Open** if it contains an 'obviously invalid' sequent (e.g. $p \vdash q$); **closed** otherwise.

From refutation display calculi to terminating tableaux

The root sequent is **invalid** iff every branch is **open**.



Conclusions and future work

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- Introduced refutation display calculi $\mathrm{D.LE}^r$ for basic LE-logics; used them to prove stuff.
- Experimented with some axiomatic extensions.

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- **On co-semidecidable logics.** Refutational proof systems as a tool to study logics that are not recursively enumerable.

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- **On co-semidecidable logics.** Refutational proof systems as a tool to study logics that are not recursively enumerable.
- **Exploring Modularity.** Characterise syntactically which axiomatic extensions of LE-logics can be captured by weakening $\mathrm{D.LE}^r$ rules alone.