On 1-variable fragments of modal predicate logics

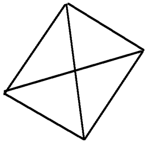
Valentin Shehtman

Higher School of Modern Mathematics, MIPT, Moscow, Russia

Higher School of Economics, Moscow, Russia

TACL 26

Krakow, July 27, 2026



Syntax-1

We consider normal 1-modal predicate logics in the signature with monadic predicate letters, without equality, constants or functions letters. So the language includes

- countably many proposition letters: $p, q, \ldots$
- countably many monadic predicate letters: $P_1, P_2, \ldots$
- the propositional connectives: $\perp, \rightarrow, \Box$
- countably many individual variables: $x, y, \ldots$
- the quantifiers: $\forall, \exists$

We use other connectives as abbreviations: $\neg, \wedge, \vee, \leftrightarrow$.
The definitions of *modal propositional formulas* and *modal predicate formulas* are standard.

Syntax-2

A *modal propositional logic* is a set $\mathbf{\Lambda}$ of modal propositional formulas such that

- $\mathbf{\Lambda}$ contains all classical propositional tautologies,
- $\mathbf{\Lambda}$ contains the axiom of $\mathbf{K}$: $\Box(p \to q) \to (\Box p \to \Box q)$,
- $\mathbf{\Lambda}$ closed under the rules:
  propositional substitution, Modus Ponens (MP), $\Box$-introduction (Nec).

We also need *bimodal* propositional formulas and logics (with $\Box$, $\blacksquare$) ; the definitions are similar.

A *modal predicate logic* is a set L of modal predicate formulas such that

- L contains all classical predicate tautologies,
- L contains the axiom of $\mathbf{K}$,
- L closed under the rules:
  predicate substitution, (MP), (Nec), $\forall$-introduction (Gen).

1-variable formulas

$\mathbf{Q\Lambda}$ is the minimal predicate extension of a modal propositional logic $\mathbf{\Lambda}$;

$$\mathbf{Q\Lambda C} := \mathbf{Q\Lambda} + \forall x \Box P(x) \to \Box \forall x P(x) \text{ (the \textit{Barcan formula})}.$$

1-variable modal predicate formulas are constructed from a single variable x and monadic predicate letters. (Proposition letters are not allowed.)

Definition

Bimodal translation of 1-variable formulas $A \mapsto A_*$ is defined recursively:

$$P_i(x)_* := p_i,$$

$$\bot_* := \bot,\ (A \to B)_* := A_* \to B_*,\ (\Box A)_* := \Box A_*,$$

$$(\forall x A)_* := \blacksquare A_*.$$

1-variable fragments-1

Definition

The *1-variable fragment* of a modal predicate logic L is the set

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is 1-variable}\}.$$

Definition

Let $\mathbf{\Lambda}$ be a (mono)modal propositional logic, $\mathbf{\Lambda} * \mathbf{S5}$ its fusion with $\mathbf{S5}$ ($=\mathbf{\Lambda}$ (in the language with $\Box$) $+$ $\mathbf{S5}$ (in the language with $\blacksquare$)),

$$\mathbf{\Lambda} \underline{\ \ \ }| \mathbf{S5} := \mathbf{\Lambda} * \mathbf{S5} + \Box\blacksquare p \to \blacksquare\Box p \ (\textit{semi-commutator}),$$

$$[\mathbf{\Lambda}, \mathbf{S5}] := \mathbf{\Lambda} * \mathbf{S5} + \Box\blacksquare p \leftrightarrow \blacksquare\Box p \ (\textit{commutator}).$$

1-variable fragments-2

Lemma 1

- $L{-}1$ is a bimodal propositional logic,
- $\mathbf{K} \lrcorner \mathbf{S5} \subseteq L{-}1$,
- $L \vdash \forall x \Box P(x) \to \Box \forall x P(x) \Rightarrow [\mathbf{K}, \mathbf{S5}] \subseteq L{-}1$.

Comment:

$$\Box \blacksquare p \to \blacksquare \Box p = (\Box \forall x P(x) \to \forall x \Box P(x))_*.$$

The latter is the converse Barcan formula provable in $\mathbf{QK}$.

MAIN PROBLEM

Given axioms of L, find axioms of $L{-}1$.

This problem seems difficult, but in some cases it has been solved. The first logic of this type was $\mathbf{QS4} - 1 = \mathbf{S4} \lrcorner \mathbf{S5}$ found by G. Fischer Servi (1977).

Horn axiomatizable logics

Definition

A *universal Horn sentence* is a classical first-order sentence in the signature $\{R, =\}$ of the form

$$\forall x \, \forall y \, \forall \bar{z} \, (\Phi(x, y, \bar{z}) \to R(x, y)),$$

where $\Phi(x, y, \bar{z})$ is a conjunction of atomic formulas.
A modal propositional formula A is *Horn* if the class of Kripke frames validating A is definable by a universal Horn sentence.

A modal propositional logic is *Horn axiomatizable* if it is axiomatized by Horn and (maybe) closed modal axioms (without proposition letters).

1-variable fragments-3

Theorem 2

If $\mathbf{\Lambda}$ is Horn axiomatizable and Kripke complete, then $\mathbf{Q\Lambda C} - 1 = [\mathbf{\Lambda}, \mathbf{S5}]$.

As we will see, this theorem follows from [Many-dimensional modal logics 2003].

Theorem 3 (Main Theorem)

If $\mathbf{\Lambda}$ is Horn axiomatizable and Kripke complete, then $\mathbf{Q\Lambda} - 1 = \mathbf{\Lambda} \;\underline{\;\;\;}|\, \mathbf{S5}$.

This theorem was first stated in [TACL19]. In this talk we will give an idea of its proof and some related results.

Products and semiproducts-1

We need only products and semiproducts with universal frames.

Definition

The *product* of Kripke frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, W_2 \times W_2)$ is the Kripke frame $F_1 \times F_2 = (W_1 \times W_2, R_h, R_v)$, where

$$\begin{array}{lcl} (x,y)R_h(x',y') & \iff & xR_1x' \quad \& \quad y = y'; \\ (x,y)R_v(x',y') & \iff & x = x'. \end{array}$$

A *semiproduct* of F_1 and F_2 is a restriction of $F_1 \times F_2$ to some $W \subseteq W_1 \times W_2$ such that $R_h(W) \subseteq W$ (i.e., W is *horizontally closed*).

If $\mathcal{C}$ and is a class of 1-frames, $\mathcal{U}$ is the class of universal frames, then we define

$$\begin{array}{lcl} \mathcal{C} \times \mathcal{U} & := & \{F_1 \times F_2 \mid F_1 \in \mathcal{C} \text{ and } F_2 \in \mathcal{U}\}, \\ \mathcal{C} \leftthreetimes \mathcal{U} & := & \{F \mid F \text{ is a semiproduct of some frames } F_1 \in \mathcal{C} \text{ and } F_2 \in \mathcal{U}\}. \end{array}$$

Products and semiproducts-2

For a modal propositional logic $\mathbf{\Lambda}$, $\mathbf{V}(\mathbf{\Lambda})$ denotes the class of all Kripke frames validating $\mathbf{\Lambda}$.
For a class of frames $\mathcal{C}$, $\mathbf{L}(\mathcal{C})$ denotes its modal logic (the set of all modal formulas valid on $\mathcal{C}$).

Definition

The *product* of 1-modal propositional logic $\mathbf{\Lambda}$ and $\mathbf{S5}$ is the 2-modal logic

$$\mathbf{\Lambda} \times \mathbf{S5} := \mathbf{L}(\mathbf{V}(\mathbf{\Lambda}) \times \mathcal{U}).$$

The *semiproduct* of $\mathbf{\Lambda}$ and $\mathbf{S5}$ is the 2-modal logic

$$\mathbf{\Lambda} \leftthreetimes \mathbf{S5} := \mathbf{L}(\mathbf{V}(\mathbf{\Lambda}) \leftthreetimes \mathcal{U}).$$

Products and semiproducts-3

Definition

Let L be a modal predicate logic, A a modal predicate formula.
A is a *consequence of L in Kripke semantics* ($L \vDash_{\mathcal{K}} A$) if for every predicate Kripke frame F

$$F \vDash L \Rightarrow F \vDash A.$$

Lemma 4

$$\widehat{L} := \{A \mid L \vDash_{\mathcal{K}} A\}$$

is the smallest Kripke complete extension (the *Kripke completion*) of L.

Products and semiproducts-4

Proposition 5

For every (mono)modal propositional logic $\mathbf{\Lambda}$:

1. $\mathbf{\Lambda} \,\underline{\ \ }|\, \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1 \subseteq \widehat{\mathbf{Q\Lambda}}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.
2. $[\mathbf{\Lambda}, \mathbf{S5}] \subseteq \mathbf{Q\Lambda C}{-}1 \subseteq \widehat{\mathbf{Q\Lambda C}}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}$.

The proof uses similarity between predicate Kripke frames and semiproducts.

Definition

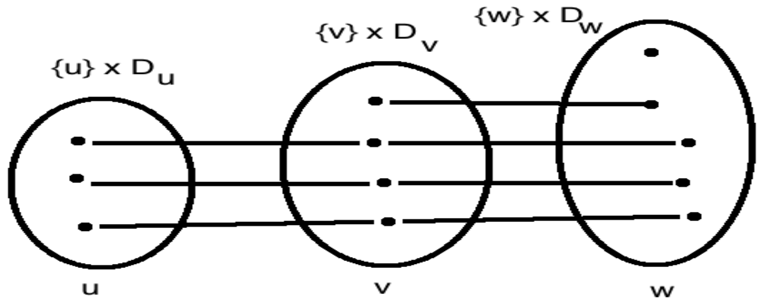
A logic $\mathbf{\Lambda}$ is

- *semiproduct-matching with* $\mathbf{S5}$ if $\mathbf{\Lambda} \leftthreetimes \mathbf{S5} = \mathbf{\Lambda} \,\underline{\ \ }|\, \mathbf{S5}$,
- *product-matching with* $\mathbf{S5}$ if $\mathbf{\Lambda} \times \mathbf{S5} = [\mathbf{\Lambda}, \mathbf{S5}]$,
- *quantifier-friendly* if $\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \,\underline{\ \ }|\, \mathbf{S5}$,
- *Barcan-friendly* if $\mathbf{Q\Lambda C}{-}1 = [\mathbf{\Lambda}, \mathbf{S5}]$.

Semiproducts and predicate Kripke frames

Every predicate Kripke frame $\mathbf{F} = (W, R, D)$ corresponds to a semiproduct of $F = (W, R)$ with a universal frame $(D^+, D^+ \times D^+)$ $\mathbf{F}_* = (W_*, R_h, R_v)$ where

- $D^+ := \bigcup\{D_u \mid u \in W\}$ is the set of all individuals,
- $W_* := \{(u, d) \mid d \in D_u\}$.



Product- and semiproduct-matching-1

Corollary 6

1. $\mathbf{\Lambda}$ is semiproduct-matching with $\mathbf{S5}$ iff
   $\mathbf{\Lambda}$ is quantifier-friendly and $\mathbf{Q\Lambda}-1$ is Kripke complete.
2. $\mathbf{\Lambda}$ is product-matching with $\mathbf{S5}$ iff
   $\mathbf{\Lambda}$ is Barcan-friendly and $\mathbf{Q\Lambda C}-1$ is Kripke complete.

The next two theorems were proved in [Many-dimensional modal logics 2003]:

Theorem 7

If $\mathbf{\Lambda}$ is Horn axiomatizable and Kripke complete, then $\mathbf{\Lambda}$ and $\mathbf{S5}$ are product-matching.

Due to Corollary 6, Theorem 7 implies Theorem 2.

Product- and semiproduct-matching-2

Theorem 8

If $\mathbf{\Lambda} \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$, then $\mathbf{\Lambda}$ and **S5** are semiproduct-matching.

By Corollary 6, Theorem 8 implies quantifier-friendlinness for these 4 logics (in particular, for **S4** noticed by Fischer Servi).

However, *the whole Theorem 7 does not transfer to semiproducts*:

$\mathbf{K5} = \mathbf{K} + \Diamond\Box p \to \Box p$ and **S5** are not semiproduct-matching [SS23].

So in the proof of Main Theorem 3 we cannot rely on semiproducts.

Instead we use simplicial semantics of modal predicate logics invented by Dmitry Skvortsov in 1990.

Ordered simplicial sets-1

$I_n := \{1, \ldots, n\},\ I_0 := \varnothing,\ \Sigma_{mn} := I_m^{I_n}$
So $\Sigma_{0n} = \{\varnothing_n\}$, where $\varnothing_n$ is the empty map, $\Sigma_{m0} = \varnothing$ for $m > 0$. Thus we have a full subcategory $\mathbf{\Sigma} \subset \mathbf{SET}$ with objects I_n, the skeleton of **FINSET**.

Definition

An ordered simplicial set is contravariant functor $\mathbf{\Sigma}^\circ \rightsquigarrow \mathbf{SET}$.

It consists of a family of sets $D = (D_n)_{n \geq 0}$ and a family of maps (transformations) $\pi = (\pi_\sigma)_{\sigma \in Mor\mathbf{\Sigma}}$ such that

- $\pi_\sigma : D_n \longrightarrow D_m$ for $\sigma \in \Sigma_{mn}$,
- $\pi_{\sigma.\tau} = \pi_\tau \cdot \pi_\sigma$, $\pi_{id_{I_n}} = id_{D_n}$.
- $\pi_{\varnothing_1} : D_1 \longrightarrow D_0$ is surjective (an additional assumption).

Ordered simplicial sets-2

Terminology:

- D_0 consists of possible worlds,
- D_1 is the set of individuals, or 0-simplices,
- D_n is the set of individuals of level n, or $(n-1)$-simplices,
- $\pi_{\varnothing_n}(s)$ is the world of an individual s,
- $D_n(u) := \pi^{-1}_{\varnothing_n}(u)$ is the domain of level n at world u.

So we have a partition $D_n = \bigcup_{u \in W} D_n(u)$. The additional assumption implies that all $\pi_{\varnothing_n}$ are surjective, i.e. the domains are non-empty.

Standard simplicial sets

Definition

A *standard ordered simplicial set of dimension* n (in another terminology, a *standard ordered simplex* of dimension n) is the following .

- $D_0 = \{*\}$ (a singleton),
- $D_1 = \{(1), \ldots, (n+1)\}$,
- $D_k = \{(a_1, \ldots, a_k) \mid 1 \leq a_1 \leq \ldots \leq a_k \leq n+1\}$,
- $\pi_\sigma(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

The intuition: $(a_1, \ldots, a_k)$ is a simplex on vertices $a_1, \ldots, a_k$, some of them may coincide. A simplex with repetitions is called *degenerate*.

Face maps and solidity

Let

$$\delta_i^n = \begin{pmatrix} 1 & \dots & i-1 & i & \dots & n-1 \\ 1 & \dots & i-1 & i+1 & \dots & n \end{pmatrix},\ \delta_1^1 = \varnothing_1.$$

The corresponding $\pi_{\delta_i^n}$ is called the i-th *hyperface map*.

In the standard case it eliminates the i-th vertex:

$\pi_{\delta_i^n} : (a_1, \dots, a_n) \mapsto \mathbf{a} - a_i$ crossing out a_i.

Definition

A *face map* in an ordered simplicial set is a composition of hyperface maps.

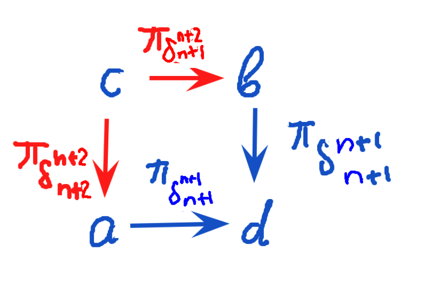
Definition

An ordered simplicial set is *solid* if

- $\pi_{\varnothing_1}$ is surjective,
- if $\pi_{\delta_{n+1}^{n+1}}(\mathbf{b}) = \pi_{\delta_{n+1}^{n+1}}(\mathbf{a})$, then there exists $\mathbf{c} \in D_{n+2}$ such that $\pi_{\delta_{n+1}^{n+2}}(\mathbf{c}) = \mathbf{b},\ \pi_{\delta_{n+2}^{n+2}}(\mathbf{c}) = \mathbf{a}$.

Solidity

The second condition is the ‘Beck – Chevalley’ property (for every $n \geq 0$):



Lemma 9

Standard ordered simplicial sets are solid.

Proof. $(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \Rightarrow \mathbf{c} = (\mathbf{a}, b_{n+1})$.

Simplicial frames-1

Let **KFrame** be the category of Kripke frames and frame morphisms $f : (W, R) \longrightarrow (W', R')$ if $f : W \longrightarrow W'$ and $\forall u \in W\ f[R(u)] = R(f(u))$.

Definition

A simplicial frame over a propositional Kripke frame $F_0 = (D_0, R_0)$ is a functor $\mathbf{\Sigma}^\circ \rightsquigarrow \mathbf{KFrame}$ satisfying the solidity conditions.

So this is a solid ordered simplicial set with accessibility relations at every level, a collection of Kripke frames $F_n = (D_n, R_n)$ and frame morphisms π_σ.

Simplicial frames-5

Definition

A valuation on $\mathbf{F}$ is a a function ξ sending proposition letters to subsets of D_0, monadic predicate letters to subsets of D_1.
A simplicial model is $(\mathbf{F}, \xi)$.
A variable assignment on $\mathbf{F}$ at $u \in W$ is $\mathbf{d}/\mathbf{x}$, where (for some n) $\mathbf{x}$ is a list of n different variables, $\mathbf{d} \in D_n(u)$.

We define the truth of a formula A at world u in a simplicial model $M = (\mathbf{F}, \xi)$ under a variable assignment $\mathbf{d}/\mathbf{x}$, where $\mathbf{x}$ contains all parameters of A.
Notation: $M, u, \mathbf{d}/\mathbf{x} \vDash A$

Simplicial frames-6

Special maps:

$\gamma_i^n : I_1 \longrightarrow I_n$, $\gamma_i^n(1) = i$, in the standard case $\pi_{\gamma_i^n}$ extracts the i-th vertex.

Definition

- $M, u, \mathbf{d}/\mathbf{x} \models q$ iff $u \in \xi(q)$,
- $M, u, \mathbf{d}/\mathbf{x} \models P(x_i)$ iff $\pi_{\gamma_i^n}(\mathbf{d}) \in \xi(P)$,
- $M, u, \mathbf{d}/\mathbf{x} \models \Box A$ iff $\forall v \in R(u)\, \forall \mathbf{e} \in D_n(v)\, (\mathbf{d} R_n \mathbf{e} \Rightarrow M, v, \mathbf{e}/\mathbf{x} \models A)$,
- $M, u, \mathbf{d}/\mathbf{x} \models \exists y A(y)$ for $y \notin \mathbf{x}$
  iff $\exists \mathbf{e} \in D_{n+1}(u)(\pi_{\delta_{n+1}^{n+1}}(\mathbf{e}) = \mathbf{d}\ \&\ M, u, \mathbf{e}/\mathbf{x}y \models A(y))$,
- $M, u, \mathbf{d}/\mathbf{x} \models \exists x_i A(x_i)$ iff $M, u, (\mathbf{d} - d_i)/(\mathbf{x} - x_i) \models \exists x_i A(x_i)$.

Other cases are clear.

Simplicial frames-7

Definition

A formula A is valid on $\mathbf{F}$ ($\mathbf{F} \vDash A$) if $\overline{\forall} A$ is true at every world in every model over $\mathbf{F}$.

A formula A is strongly valid on $\mathbf{F}$ ($\mathbf{F} \vDash^{+} A$) if all its substitution instances are valid.

Proposition 10 (Skvortsov, 1990; cf. [SS93])

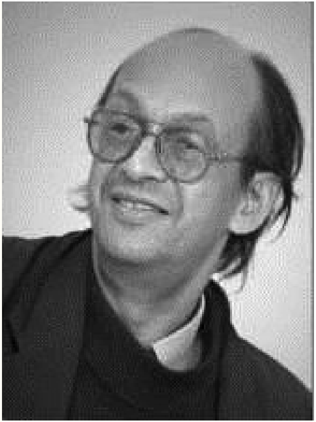
For a propositional A,

$$\mathbf{F} \vDash^{+} A \text{ iff } \forall n\, F_n \vDash A.$$

Soundness theorem 11 (Skvortsov, 1990; cf. [SS93])

Let $\mathbf{F}$ be a simplicial frame. Then $\mathbf{L}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash^{+} A\}$ is a modal predicate logic.

Simplicial frames-8



Dmitry Skvortsov (1952-2025):

"I don't understand how I came up with all this stuff"

Main theorem: the idea of proof-1

Given $A_* \notin \mathbf{\Lambda}_1 := \mathbf{\Lambda} \,\underline{\ \ }|\, \mathbf{S5}$, we show that $A_* \notin \mathbf{Q\Lambda}{-}1$, i.e., $A \notin \mathbf{Q\Lambda}$.

We refute A in a simplicial $\mathbf{\Lambda}$-frame.

$\mathbf{\Lambda}$ is Kripke complete and elementary, so by standard theorems from modal logic we have:

Lemma 12

$\mathbf{\Lambda}_1$ is canonical, thus Kripke complete.

If $(W, R, S) \vDash \mathbf{\Lambda}_1$ and $(W, R, S) \nvDash A_*$, then $SR \subseteq RS$ and S is an equivalence.

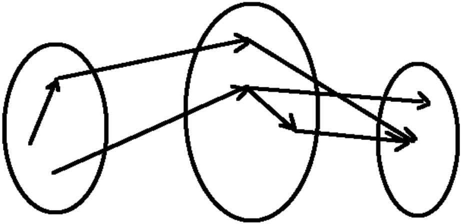
This is a 'Kripke bundle' giving the first two levels of the sought simplicial frame:

$(D_1, R_1) = (W, R)$ (the frame of individuals),

$(D_0, R_0) = (W/S, R_0)$ (the frame of possible worlds that are S-clusters), with $S(a) R_0 S(b) \iff aRSb$, $\pi_{\varnothing 1}(a) = S(a)$.

Then $(D_1, R_1) \vDash \mathbf{\Lambda}$, and $(D_0, R_0) \vDash \mathbf{\Lambda}$ as a p-morphic image.

Constructing a Kripke bundle



F=(W,R,S)

Degeneracy and permutation maps

$$\sigma_i^n = \begin{pmatrix} 1 & \dots & i & i+1 & i+2 & \dots & n+1 \\ 1 & \dots & i & i & i+1 & \dots & n \end{pmatrix},$$

$\pi_{\sigma_i^n} : X_n \longrightarrow X_{n+1}$ are called *elementary degeneracy* maps.
In a standard simplex:
$\pi_{\sigma_i^n}(a_1, \dots, a_n) = (a_1, \dots, a_i, a_i \dots, a_n)$.

Definition

- *Permutation maps* are π_τ for permutations τ.
- *Face maps* are compositions of hyperface maps.
- *Degeneracy maps* are compositions of elementary degeneracy maps and, maybe, permutation maps.

Lemma 13

Every transformation is a composition of a degeneracy and a face map.

Degenerate simplices

Definition

A *degenerate simplex* is an image of a degeneracy map.

Lemma 14 (ordered Eilenberg – Zilber's lemma)

Every degenerate simplex is obtained by a degeneracy map from a non-degenerate simplex, unique up to permutation.

Groupoids and their nerves-1

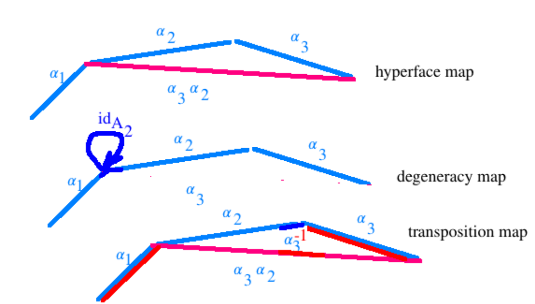
Definition

A *groupoid* is a small category, in which all arrows are invertible.

In a groupoid $\mathbf{C}$ every $\mathbf{C}(A, A)$ is a group, all these groups are isomorphic.

Examples: fundamental groupoids of topological spaces (Poincaré).

The *components* of $\mathbf{C}$ are the equivalence classes of the relation $\mathbf{C}(A, B) \neq \varnothing\}$ on $Ob\mathbf{C}$.



Groupoids and their nerves-2

Definition (≈ Grothendieck)

The *(ordered) nerve* of a groupoid $\mathbf{C}$ is the ordered simplicial set $\mathcal{N}(\mathbf{C})$ where:

- X_n is the set of all paths in $\mathbf{C}$ with n points: $(A_1, \alpha_1, \ldots, A_n)$, $X_1 = Ob\mathbf{C}$, X_0=the set of components of $\mathbf{C}$,
- $\pi_{\delta_1^n}$ eliminates A_1, α_1,
- $\pi_{\delta_n^n}$ eliminates α_{n-1}, A_n,
- otherwise $\pi_{\delta_i^n}$ replaces $\alpha_{i-1}, A_i, \alpha_i$ with $\alpha_i \alpha_{i-1}$,
- $\pi_{\sigma_i^n}$ replaces A_i with A_i, id_{A_i}, A_i,
- π_{τ_1} replaces $A_1, \alpha_1, A_2, \alpha_2$ with $A_2, \alpha_1^{-1}, A_1, \alpha_2 \alpha_1$,
- $\pi_{\tau_{n-1}}$ replaces $\alpha_{n-2}, A_{n-1}, \alpha_{n-1}, A_n$ with $\alpha_{n-1}\alpha_{n-2}, A_n, \alpha_{n-1}^{-1}, A_{n-1}$,
- otherwise π_{τ_i} replaces $\alpha_{i-1}, A_i, \alpha_i, A_{i+1}, \alpha_{i+1}$ with $\alpha_i \alpha_{i-1}, A_{i+1}, \alpha_i^{-1}, A_i, \alpha_{i+1}\alpha_i$.

Main theorem: the idea of proof-2

We have a Kripke bundle $\Phi = (W, R, S) \vDash \mathbf{\Lambda}_1$ refuting A_*, and we extend it to a simplicial frame $\mathbf{F}$.

1. We define a groupoid $\mathbf{C}$, with $Ob\mathbf{C} = W = D_1$,

$$\mathbf{C}(a,b) = \begin{cases} \{(a,b)\} \times \mathbf{Z} & \text{if } aSb, \\ \varnothing & \text{otherwise.} \end{cases}$$

The composition:

$$(b,c,n) \cdot (a,b,m) := (a,c,m+n).$$

2. We define relations on $\mathcal{N}(\mathbf{C})$.

2.1. For non-degenerate simplices $\mathbf{a} = (a_1, \alpha_1, \dots, a_n)$, $\mathbf{b} = (b_1, \alpha_1, \dots, b_n)$

$$\mathbf{a} R_n \mathbf{b} \iff \forall i\, a_i R b_i.$$

2.2 For degenerate simplices: if π_σ is a degeneracy map, then

$$\pi_\sigma(\mathbf{a}) R_m \pi_\sigma(\mathbf{b}) \iff \mathbf{a} R_n \mathbf{b}.$$

Main theorem: the idea of proof-3

Lemma 15 (Main Lemma)

- $\mathbf{F}$ is a simplicial frame.
- $\mathbf{F} \models^{+} \mathbf{\Lambda}$.

Remarks on the proof.

1. The nerve of every groupoid is an ordered simplicial set.
2. Infinity of $\mathbf{Z}$ guarantees that $\pi_{\delta_i^n}$ are morphisms.
3. Horn formulas lift from (W, R) to all frames (D_n, R_n).

Further results-1

Corollary 16

If $\mathbf{\Lambda}$ is Horn axiomatizable and $\mathbf{Q\Lambda}$ is Kripke complete, then $\mathbf{\Lambda}$ is semiproduct-matching with $\mathbf{S5}$.

This implies Theorem 8 and a bit more: for $\mathbf{\Lambda}$ axiomatized by formulas of the form $\Box p \to \Box^n p$ ("one-way PTC logics" from [QNL]).

Lemma 17

The Barcan formula lifts from Φ to $\mathbf{F}$.

Theorem 18

If $\mathbf{\Lambda}$ is obtained from $\mathbf{K4}$ or $\mathbf{S4}$ by adding some axioms of the form $\Box\Diamond p \to \Diamond\Box p$ (McKinsey), $\Diamond\Box p \to \Box\Diamond p$ (confluence), or/and closed, then $\mathbf{\Lambda}$ is quantifier-friendly and Barcan-friendly.

Further results-2

Corollary 19

The logics **K4.2**, **S4.2** are semiproduct-matching with **S5**.

Follows from Theorem 18 and Kripke completeness of **QK4.2**, **QS4.2** [Density18], [Corsi & Ghilardi 1989].

References-1

[Fischer Servi 77] G. Fischer Servi. On modal logic with an intuitionistic base. Studia Logica, 36:141-149, 1977.

[Many-dimensional logics 2003] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-Dimensional Modal Logics: Theory and Applications, Elsevier, 2003.

[Products 1998] D. Gabbay, V. S. Products of modal logics, Part 1. Logic Journal of the IGPL, 6(1):73–146, 1998.

[TACL19] V. S. Simplicial semantics and one-variable fragments of modal predicate logics. In: TACL 2019, Abstracts, pp. 172–173, Nice, 2019.

[SS23] V. S., D. Shkatov. Semiproducts, products, and modal predicate logics: some examples. Doklady Mathematics, 108 (2), 411—418, 2023.

[QNL] D. Gabbay, V. S., D. Skvortsov. Quantification in Nonclassical Logic, v.1. Elsevier, 2009.

References-2

[Density18] V.S. On Kripke completeness of some modal predicate logics with the density axiom. Advances in Modal Logic, v. 12, College Publications, 2018, pp. 559–576.
[SS93] D. Skvortsov, V.S. Maximal Kripke-type semantics for modal and superintuitionistic predicate logics. Annals of Pure and Applied Logic, 63, 69–101, 1993.

[Corsi & Ghilardi 1989] G. Corsi, S. Ghilardi. Directed frames. Archive for Mathematical Logic, v.29, 53-67, 1989.

THANK YOU!