On lifting maps to the Wallman compactification

Elena Pozzan

University of Turin

28/07/2026

Joint work with Mai Gehrke and Matteo Viale

Topological spaces from families of filters

Given a family of filters $\mathcal{A}$ on a lattice P, we can define a topological space $(\mathcal{A}, P^*)$, where the topology is generated by the sets:

$$\mathcal{N}_p^{\mathcal{A}} = \{F \in \mathcal{A} : p \in F\}.$$

The topological properties of the space of filters are related to the algebraic properties of the family of filters and of the lattice.

Examples:

- in Stone duality, this leads to a duality between spectral spaces and bounded distributive lattices;
- in the Omega-Point duality, this leads to a duality between sober spaces and spatial frames.

Idea

Theorem (Wallman)

Let P be a bounded distributive lattice and

$$\mathrm{W}(P) = \{F \subseteq P : F \text{ is a minimal prime filter on } P\}$$

Then $(\mathrm{W}(P), P^*)$ is a compact T_1 space.
Moreover, if P is normal then $(\mathrm{W}(P), P^*)$ is a compact T_2 space.

Wallman compactification

Definition

Let (X, τ) be a T_1 space, then the **Wallman compactification** of X is $(\mathsf{W}(\tau), \tau^*)$. We denote this space by $(\mathsf{W}(X), \tau^*)$.

The canonical map

$$\eta_X : X \to \mathsf{W}(X), \qquad x \mapsto F_x = \{U \in \tau \mid x \in U\},$$

is a dense embedding.

If X is **normal**, then its Wallman compactification coincides with its **Stone–Čech compactification**.

Stone–Čech compactification

The Stone–Čech compactification is the reflector associated with the full inclusion

$$\mathsf{Cpct\text{-}T_2} \hookrightarrow \mathsf{Top},$$

that is,

$$\beta : \mathsf{Top} \to \mathsf{Cpct\text{-}T_2}.$$

When restricted to Tych, this reflection is an epireflection: for every $X \in \mathsf{Tych}$, the unit

$$\iota_X : X \hookrightarrow \beta X$$

is an epimorphism in Tych.

Herrlich's problem

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Find a category whose objects are the T_1 spaces and whose morphisms are continuous maps, on which the Wallman compactification yields an epireflection.

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In this talk, we focus on two classes of maps that solve Herrlich's problem:

$$\mathcal{WO} \qquad \text{and} \qquad \mathcal{WC}.$$

What an extension must do

Let

$$f : (X, Q) \to (Y, P)$$

be a continuous map between T_1 spaces and let

$$i_f : P \to Q, \qquad V \mapsto f^{-1}[V].$$

Suppose that f has a continuous Wallman extension

$$g : (\mathrm{W}(X), Q^*) \to (\mathrm{W}(Y), P^*).$$

For every point $F \in \mathrm{W}(X)$, the value $g(F)$ is forced to satisfy

$$g(F) \subseteq i_f^{-1}[F].$$

Therefore, the extension problem is controlled by the minimal prime filters contained in $i_f^{-1}[F]$.

This suggests studying the problem algebraically.

The relation $\prec_f$

Let

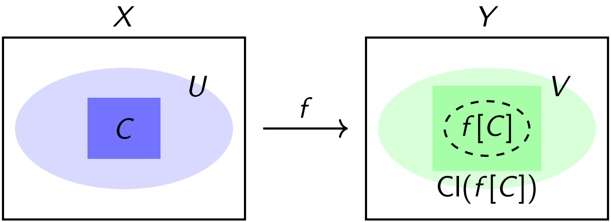
$$f : (X, Q) \to (Y, P)$$

be a continuous map between T_1 spaces. For $U \in Q$ and $V \in P$, we write

$$U \prec_f V$$

if for every closed subset $C \subseteq U$,

$$\mathrm{Cl}_Y(f[C]) \subseteq V.$$



$\mathcal{WO}$-maps

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Definition

We say that f is a $\mathcal{WO}$-**map** if for every finite open cover $\mathcal{V}$ of Y, there exists a finite open cover $\mathcal{U}$ of X such that:

$$\forall U \in \mathcal{U} \;\; \exists V \in \mathcal{V} \quad \text{with} \quad U \prec_f V.$$

The algebraic relation $\prec_i$

Let

$$i : P \to Q$$

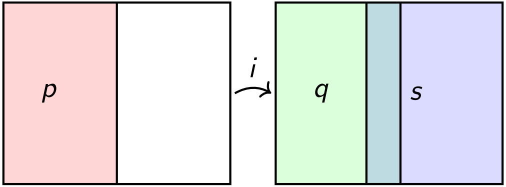
be a morphism of bounded distributive lattices.

For $p \in P$ and $q \in Q$, we write

$$q \prec_i p$$

if, for every $s \in Q$,

$$s \vee q = 1_Q \implies \exists s' \in P(i(s') \leq s \text{ and } s' \vee p = 1_P)$$



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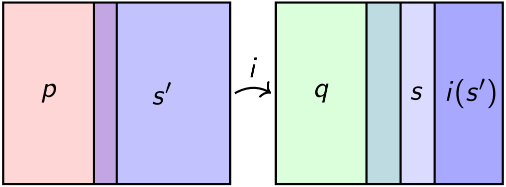

$\mathcal{WO}$-morphisms

Definition

A morphism of bounded lattices $i : P \to Q$ is a $\mathcal{WO}$-**morphism** if for every finite cover $\mathcal{P}$ of P there exists a finite cover $\mathcal{Q}$ of Q such that for every $q \in \mathcal{Q}$, there exists $p \in \mathcal{P}$ with $q \prec_i p$.

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- Every $\mathcal{WO}$-map $f : (X, Q) \to (Y, P)$ induces a $\mathcal{WO}$-morphism

$$i_f : P \to Q, \qquad V \mapsto f^{-1}[V].$$

- Every $\mathcal{WO}$-morphism $i : P \to Q$ induces a $\mathcal{WO}$-map. For every $F \in \mathrm{W}(Q)$, the filter $i^{-1}[F]$ contains a **unique minimal prime filter** on P, denoted by $\mathrm{W}(i)(F)$. Hence one obtains a continuous map

$$\mathrm{W}(i) : \mathrm{W}(Q) \to \mathrm{W}(P), \qquad F \mapsto \mathrm{W}(i)(F).$$

The $\mathcal{WO}$-adjunction

Let

$$\begin{aligned}
\mathcal{L} : (\mathsf{T}_{1\mathcal{WO}})^{\mathrm{op}} &\longrightarrow \mathsf{DL}_{\mathcal{WO}} \\
(X, Q) &\longmapsto Q \\
\big(f : (X, Q) \to (Y, P)\big) &\longmapsto \big(i_f : P \to Q\big)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R} : (\mathsf{DL}_{\mathcal{WO}})^{\mathrm{op}} &\longrightarrow \mathsf{T}_{1\mathcal{WO}} \\
P &\longmapsto (\mathsf{W}(P), P^*) \\
\big(i : P \to Q\big) &\longmapsto \big(\mathsf{W}(i) : (\mathsf{W}(Q), Q^*) \to (\mathsf{W}(P), P^*)\big),
\end{aligned}$$

Theorem (Harris/P.-Viale)

The pair $(\mathcal{L}, \mathcal{R})$ forms a contravariant adjunction.

A limitation of $\mathcal{WO}$-maps

The class of $\mathcal{WO}$-maps solves Herrlich's problem but has several limitations:

- not every map admitting a unique continuous Wallman extension is a $\mathcal{WO}$-map;
- even between compact T_1 spaces, there are continuous maps that are not $\mathcal{WO}$-maps;
- the definition of $\mathcal{WO}$-maps may seem somewhat ad hoc.

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This suggests looking at a more rigid but conceptually more natural class of maps.

$$\boxed{\text{Require the Wallman extension to be closed.}}$$

$\mathcal{WC}$-maps

Definition (Harris)

A continuous map between T_1 spaces is a $\mathcal{WC}$-**map** if it admits a (unique) continuous and *closed* Wallman extension.

Problem (Harris)

Give an internal characterization of $\mathcal{WC}$-maps.

$\mathcal{WC}$-morphisms

Definition

A morphism of bounded distributive lattices $i : P \to Q$ is a $\mathcal{WC}$-**morphism** if:

1. whenever $p \vee p' = 1_P$, there exist $q, q' \in Q$ such that
$$q \prec_i p, \qquad q' \prec_i p', \qquad q \vee q' = 1_Q.$$
2. the relation $\prec_i$ is closed under finite joins in the first coordinate:
$$q \prec_i p,\ q' \prec_i p \quad \Longrightarrow \quad q \vee q' \prec_i p.$$

Internal characterization of $\mathcal{WC}$ maps

- If $i : P \to Q$ is a $\mathcal{WC}$-morphism, then $\mathrm{W}(i) : \mathrm{W}(Q) \to \mathrm{W}(P)$ is a continuous closed map;
- If $f : X \to Y$ is a $\mathcal{WC}$-map, then i_f is a $\mathcal{WC}$-morphism.

Theorem (P.-Viale)

There exists a contravariant adjunction between $\mathrm{T}_{1\mathcal{WC}}$ and $\mathrm{DL}_{\mathcal{WC}}$.

Corollary

Let $f : (X, Q) \to (Y, P)$ be a continuous map between T_1 spaces. Then f is a $\mathcal{WC}$-map if and only if $i_f : P \to Q$ is a $\mathcal{WC}$-morphism.

Thank You!

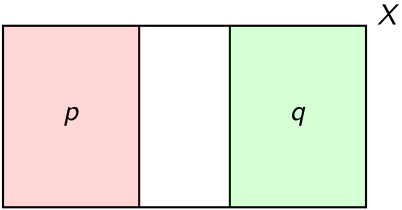
Duality

The previous adjunction restricts to a duality for compact T_1 spaces. To describe the objects on the algebraic side, we need two additional notions: subfitness and compactness.

Subfit lattices

Definition

A bounded distributive lattice P is **subfit** if for every pair of distinct elements $p, q \in P$ there exists $z \in P$ such that $p \vee z = 1_P$ and $q \vee z \neq 1_P$ or vice versa.



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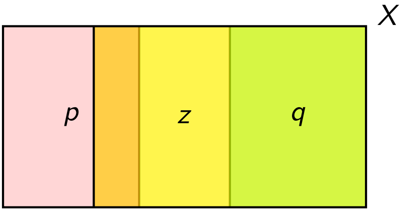

Closed subfit morphisms

Definition

A morphism of bounded distributive lattices $i : P \to Q$ is **closed subfit** if:

$$i(p) \vee q = 1_Q \implies \exists p' \in P(p \vee p' = 1_P \text{ and } i(p') \leq q).$$

Let $i : P \to Q$ be a morphism between bounded distributive lattices.

- If i is closed subfit, then i is a $\mathcal{WC}$-morphism;
- If P and Q are complete, compact and subfit, then i is closed subfit if and only if it is a $\mathcal{WC}$-morphism.

Closed subfit morphisms are the algebraic counterparts of closed continuous maps between compact T_1 spaces.

The compact T_1 duality

Let

$$\mathsf{Cpct\text{-}T1}_{\mathsf{closed}}$$

be the category of compact T_1 spaces and closed continuous maps.

Let

$$\mathsf{Cpct\text{-}Sbf\text{-}Fr}_{\mathsf{clsbf}}$$

be the category of compact and subfit frames and closed subfit morphisms.

Theorem (Gehrke-P.-Viale)

The previous functors restrict to a duality

$$\mathsf{Cpct\text{-}T1}_{\mathsf{closed}} \simeq (\mathsf{Cpct\text{-}Sbf\text{-}Fr}_{\mathsf{clsbf}})^{\mathrm{op}}.$$

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