The compactness theorem for topos-valued models

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coherent categories $\supseteq$ Boolean coh. cat.

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models: coherent functors $\mathcal{C} \to \mathbf{Set}$

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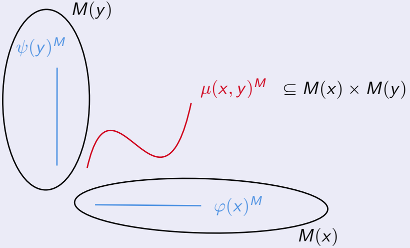
frames are injective in $\mathbf{Lat}_\wedge$	Grothendieck toposes are injective in $\mathbf{Lex}$

Coherent categories

$Def(M)$: cat. of $\varnothing$-def. sets



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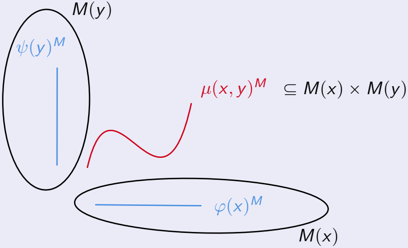

$M \models T \rightsquigarrow$ category $\mathcal{C}_T$ of formulas,

$ev_M : \mathcal{C}_T \to Def(M) \subseteq \mathbf{Set}$ well-def,

def. limits and colimits live in $\mathcal{C}_T$.

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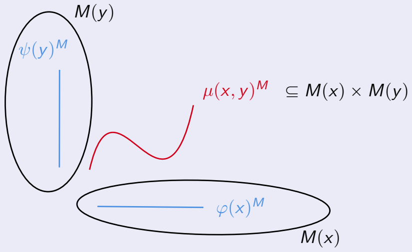

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\begin{tikzcd}
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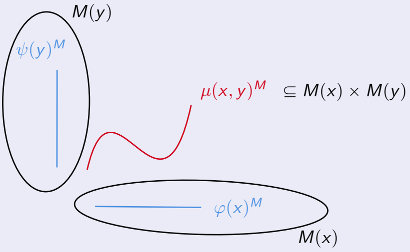

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(Boolean) coherent category

- finite limits
- pb-stab. eff. epi - mono factorization
- pb-stab. finite unions
- (subobjects are complemented)

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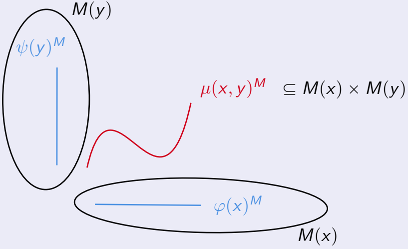

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Type spaces

$\mathcal{C} \xrightarrow{Sub_{\mathcal{C}}} \mathbf{DLat}^{op} \xrightarrow{Spec} \mathbf{Spec} \subseteq \mathbf{Set}$

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Grothendieck toposes:

$$Sh(X),\ \ \mathbf{Set}^{\mathcal{D}},\ \ Sh(B, \tau_{can}),\ \ Sh(\mathcal{C}, \tau_{coh}),\ \ \mathbb{S}_\omega(\mathcal{A}) = [\mathcal{A}, \mathbf{Set}]_{\text{filt. colim}}$$

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Internal models:

sorts	$\rightsquigarrow$	objects,
funct. symb.	$\rightsquigarrow$	arrows,
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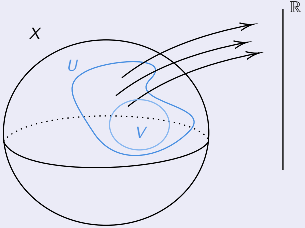
Motivation:

- Model theory of such things.
- Nice class of accessible categories: some compactness remains.

A sheaf of structures

Example: $\mathit{Cont}(-, \mathbb{R})$



Definition

L signature. A sheaf of L-structures on X is a functor

$$M : \mathit{Open}(X)^{op} \to \mathbf{Str}_L$$

s.t. the gluing condition holds.

A sheaf of structures

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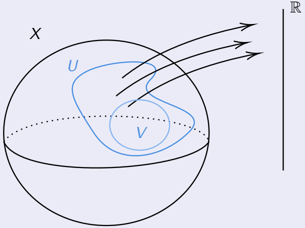

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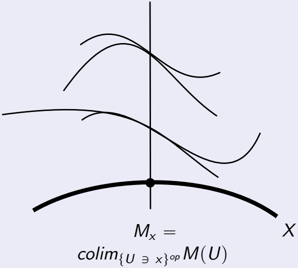
Stalk at x.

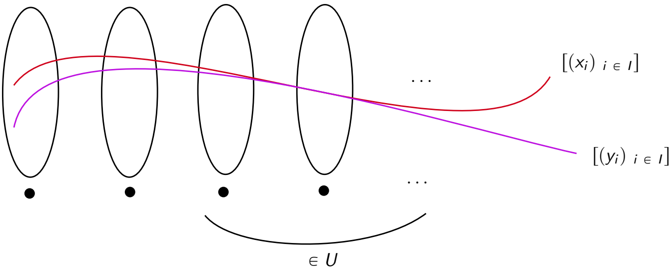


Definition

$T \subseteq L^g_{\omega\omega}$. A sheaf of L-structures $M : Open(X)^{op} \to \mathbf{Str}_L$ is a model of T, if for each $x \in X$ we have $M_x \models T$.

Ultraproducts.

I is a set, $U \subseteq \mathcal{P}(I)$ is an ultrafilter. $\prod_i X_i \big/ U$ is given by



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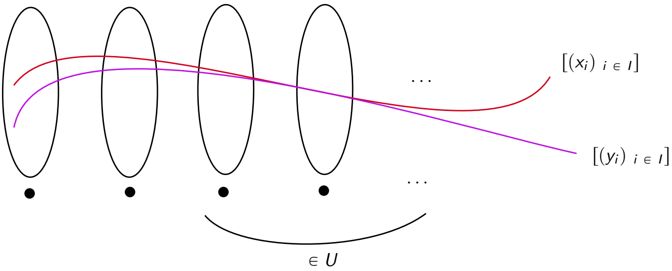

This is a directed colimit of products:

$$
\begin{tikzcd}
 & & & \prod_{i\in I} X_i \big/ U = colim_{J\in U} \prod_{i\in J} X_i \\
\prod_{i\in J} X_i \arrow[r, "\langle\pi_i\rangle_{i\in J\cap J'}"] \arrow[urrr, bend left] & \prod_{i\in J\cap J'} X_i \arrow[r] \arrow[urr, bend left] & \cdots & \\
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Remark

$\mathbf{Coh}(\mathcal{C}, \mathbf{Set}^{\mathcal{D}}) \subseteq [\mathcal{C}, \mathbf{Set}^{\mathcal{D}}]$ *is closed under ultraproducts:*
$M : \mathcal{C} \to \mathbf{Set}^{\mathcal{D}}$ *is coherent iff each* $M_d = ev_d \circ M$ *is coherent,*
and ultraproducts are computed pointwise.

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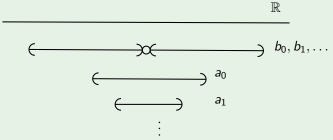
Question: Is it true that $\mathbf{Coh}(\mathcal{C}, Sh(X)) \subseteq [\mathcal{C}, Sh(X)]$ is closed under ultraproducts? (No.)

Example 1

$$\mathcal{C} \;=\; \begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & a \vee b \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \arrow[rr, bend left, "M_i"] & & Sh(\mathbb{R}) \\ & a \wedge b \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$



$$\bigcup_{J \in U} \bigcap_{i \in J} (a_i \cup b_i) \supsetneqq \bigcup_{J \in U} \bigcap_{i \in J} a_i \;\cup\; \bigcup_{J \in U} \bigcap_{i \in J} b_i$$

where $M_i(a) = a_i \in Sub_{Sh(X)}(1) = Open(\mathbb{R})$ and $M_i(b) = b_i \in Open(\mathbb{R})$.

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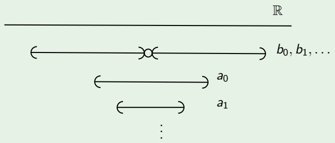

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Motto for Example 1

In $Sh(X)$, infinite intersections do not interact well with finite unions.

Example 2

- B is an arbitrary infinite Boolean algebra, $U \subseteq \mathcal{P}(\omega)$ is non-principal.
- $M : \mathbf{Set}_{fin} \to Sh(B, \tau_{coh})$ is the unique coherent functor.
- In particular $M(1 \sqcup 1) = \mathbf{1} \sqcup \mathbf{1}$.
  Note that $\mathbf{1} \sqcup \mathbf{1}(\top) = Sh(B)(\mathbf{1}, \mathbf{1} \sqcup \mathbf{1}) = Sub^{\neg}_{Sh(B)}(\mathbf{1}) = B$.
- $Sh(Spec(B)) \subseteq \mathbf{Set}^{Spec(B)^{op}}$ is closed under limits and filt. colim.
- Hence ultraproducts are pointwise. In particular
  $({}^{M^\omega}\!/_{U})(1 \sqcup 1)(\top) = {}^{(M(1 \sqcup 1)(\top))^\omega}\!/_{U} = {}^{B^\omega}\!/_{U} \neq B$.

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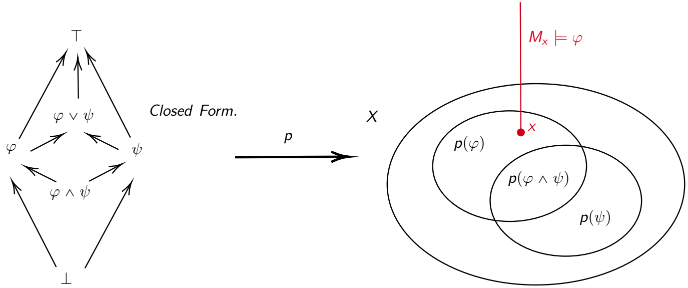
Motto for Example 2

The ultraproduct of parametrized families (each living over $X = Spec(B)$) is parametrized over a larger space.

Realizing types.

Can we prescribe which statements to hold in which stalks?



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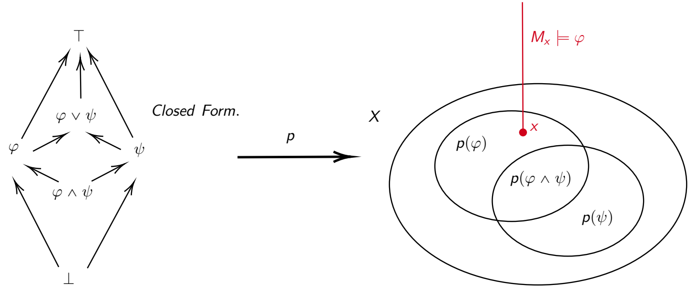

Definition: $\mathcal{E}$ can realize types, if

for any small coherent cat. $\mathcal{C}$ and coherent functor $p : Sub_{\mathcal{C}}(1) \to \mathcal{E}$, there is a coherent lift:

$$
\begin{tikzcd}
Sub_{\mathcal{C}}(1) \arrow[r, "p"] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow"] & \mathcal{E} \\
\mathcal{C} \arrow[ur, "M"'] &
\end{tikzcd}
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Remark

If $Sh(Sub_{\mathcal{E}}(1), \tau_{coh})$ can realize types, then $\mathcal{E}$ can realize types.

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Proof.

$$\begin{tikzcd}
Sub_{\mathcal{C}}(1) \arrow[r, "p_0"] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow"] & Sub_{\mathcal{E}}(1) \arrow[r, phantom, "\subseteq"] & Sh(Sub_{\mathcal{E}}(1), \tau_{coh}) \arrow[r] & \mathcal{E} \\
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$\square$

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\end{tikzcd}$$

□

So it suffices to treat toposes of the form $Sh(L, \tau_{coh})$ where L is a frame.

(frame = complete distributive lattice with $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$)

Theorem

Let L be a distributive lattice and let Q be a well-founded forest. Assume that L is a lax retract of the lattice of upsets $(Up(Q), \subseteq)$. Then $Sh(L, \tau_{coh})$ can realize types.

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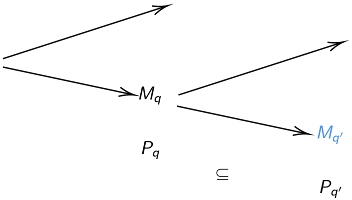
proof.

I. Q-indexed families: **Set**^{Q} can realize types.



II. prime products: $Sh(Up(Q))$ can realize types.

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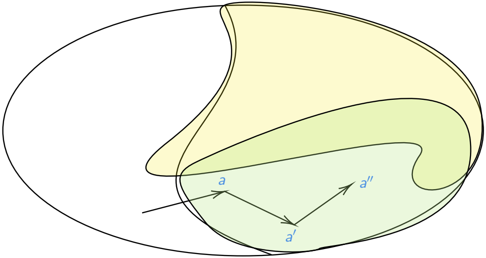
Coherent:

- finite limits ✓
- effective epis: already preserved on opens, i.e. $\lim_{q\in J} M_q(x) \twoheadrightarrow \lim_{q\in J} M_q(y)$ eff. epi (Q w-f forest!)
- finite unions:



$(a_i)_i \in \lim_{q\in J} M_q(u \cup v)$

$\Rightarrow$

$[(a_i)_i] \in colim_J \quad \lim_{q\in J} M_q(u)$

or

$[(a_i)_i] \in colim_J \quad \lim_{q\in J} M_q(u)$

III. lax retracts: $Sh(L)$ can realize types.

$$
\begin{tikzcd}
Sub_{\mathcal{C}}(1) \arrow[r, "p"] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow"] & Sh(L) \arrow[r, "j^*"] \arrow[rr, bend left=60, equal] \arrow[rr, phantom, bend left=25, "\Downarrow"] & Sh(Up(Q)) \arrow[r, "r^*"] & Sh(L) \\
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$$

$\square$

Two missing ingredients for

"Every Grothendieck topos can realize types".

Let L be a (bounded) distributive lattice. Its *canonical extension* is

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which sends x to the up-set $\{P : x \in P\}$.

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If L is De Morgan, then yes:

$$\begin{tikzcd}
L \arrow[rrrr, equal] \arrow[dr] & & & & L \\
& L^{\neg} = B \arrow[rr, equal] \arrow[dr, hook] & {} \arrow[u, phantom, "\rotatebox{90}{$\leq$}"] \arrow[d, phantom, "="] & B \arrow[ur, hook] & \\
& & 2^{Spec(B)} \arrow[ur, "r"] & &
\end{tikzcd}$$

Corollary: every De Morgan topos can realize types

Let L be a (bounded) distributive lattice. Its *canonical extension* is

$$L \hookrightarrow (Up(Spec(L)), \subseteq)$$

which sends x to the up-set $\{P : x \in P\}$.

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Let L be a frame. Does the canonical extension lax-split?

If L is De Morgan, then yes:

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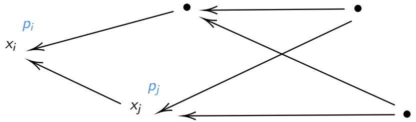
If L is completely distributive, then yes:

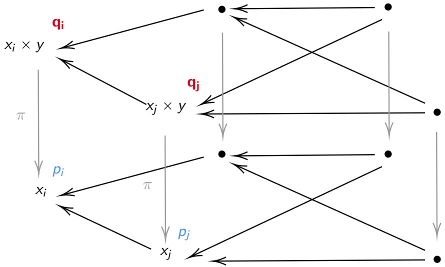
The canonical ext. is the free completely distributive lattice over L.

$\mathcal{C}$ is a small coherent category, $(Q, \leqslant)$ is a poset.

Given a Q-indexed diagram in $Spec(\mathcal{C})$



and $y \twoheadrightarrow 1$, can we find lifts



making every map continuous?

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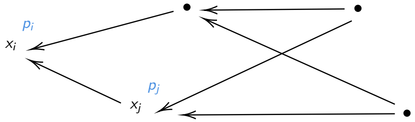

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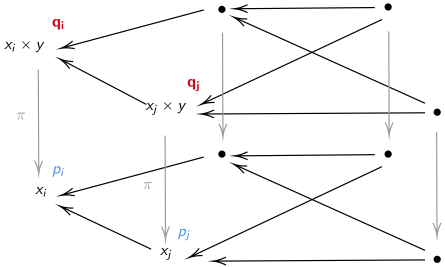

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Remark

If Q is a well-founded forest: yes.

Thank you,
and grant ERC 101170308.