A first-order extension of Multilingual Sequent Calculus

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Content

Some background

Multilingual Sequent Calculus

Some background: logic of observable properties

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- In [1], Samson Abramsky developed the program of domain theory in logical form.
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- In his PhD thesis [3], Mathias Kegelmann extended Abramsky's program to continuous domains.
- To achieve this, he introduced the *Multi-Lingual Sequent Calculus* (**MLS**).
- **MLS** is forms a category admits a representation in terms of stably compact spaces.

Stably Compact Spaces

Definition

A topological space X is *stably compact* if it is:

- sober,
- locally compact,
- stable: finite intersections of compact saturated sets are compact.

MLS

A *token algebra* is an algebra for the signature $\langle \wedge, \vee, \top, \bot \rangle$, where $\wedge$ and $\vee$ are binary operations and $\top$ and $\bot$ are constants, as usual [[3],[4]].
A *consequence relation* from a token algebra L to a token algebra M is a binary relation $\vdash \subseteq \mathcal{B}_{\mathrm{fin}}(L) \times \mathcal{B}_{\mathrm{fin}}(M)$ obeying the following rules of Gentzen's calculus:

MLS

$$\frac{}{\bot \vdash} (L\bot) \qquad \frac{\mathcal{S} \vdash \mathcal{T}}{\mathcal{S} \vdash \mathcal{T}, \bot} (R\bot)$$

$$\frac{\mathcal{S} \vdash \mathcal{T}}{\top, \mathcal{S} \vdash \mathcal{T}} (L\bot) \qquad \frac{}{\vdash \top} (R\bot)$$

$$\frac{\phi, \psi, \mathcal{S} \vdash \mathcal{T}}{\phi \wedge \psi, \mathcal{S} \vdash \mathcal{T}} (\wedge\mathrm{L}) \qquad \frac{\mathcal{S} \vdash \mathcal{T}, \phi \qquad \mathcal{S} \vdash \mathcal{T}, \psi}{\mathcal{S} \vdash \mathcal{T}, \phi \wedge \psi} (\wedge\mathrm{R})$$

$$\frac{\phi, \mathcal{S} \vdash \mathcal{T} \qquad \psi, \mathcal{S} \vdash \mathcal{T}}{\phi \vee \psi, \mathcal{S} \vdash \mathcal{T}} (\vee\mathrm{L}) \qquad \frac{\mathcal{S} \vdash \mathcal{T}, \phi, \psi}{\mathcal{S} \vdash \mathcal{T}, \phi \vee \psi} (\vee\mathrm{R})$$

The backwards rules are not present in the usual sequent calculus since there they are consequences of the identity and the cut rule [3].

MLS

and the *Weakening*:

$$\frac{\mathcal{S} \vdash \mathcal{T}}{\mathcal{S}', \mathcal{S} \vdash \mathcal{T}, \mathcal{T}'} \text{(W)}$$

Contraction and *Exchange* are implicit since $\vdash$ is a relation between finite sets.
For two consequences relations $\vdash : L \longrightarrow M$ and $\vdash' : M \longrightarrow N$ define the composition $\vdash ; \vdash'$ by.

$$\frac{\mathcal{S} \vdash \alpha \qquad \alpha \vdash' \mathcal{T}}{\mathcal{S} \vdash ; \vdash' \mathcal{T}} \text{(CUT')}$$

MLS

Definition

A *stable calculus* is a token algebra L equipped with a consequence relation $\Vdash$ from L to L so that the following hold:

1. it is closed under CUT
$$\frac{\mathcal{S}_0 \Vdash_L \mathcal{T}_0, \alpha \qquad \alpha, \mathcal{S}_1 \Vdash_L \mathcal{T}_1}{\mathcal{S}_0, \mathcal{S}_1 \Vdash_L \mathcal{T}_0, \mathcal{T}_1} \text{ (CUT)}$$
2. $\Vdash_L$ enjoys sequential cut decomposition: if $\mathcal{S}_0, \mathcal{S}_1 \Vdash_L \mathcal{T}_0, \mathcal{T}_1$, where either $\mathcal{S}_1 = \varnothing$ or $\mathcal{T}_0 = \varnothing$, then there exists a formula ϕ such that $\mathcal{S}_0 \Vdash_L \mathcal{T}_0, \phi$ and $\phi, \mathcal{S}_1 \Vdash_L \mathcal{T}_1$.
3. the logical rules for $\wedge$ and $\vee$ hold in reverse.

A *compatible consequence relation with stable calculi* $(L, \Vdash_L)$ and $(M, \Vdash_M)$, is a consequence relation $\vdash$ such that $\Vdash_L ; \vdash = \vdash = \vdash ; \Vdash_M$.

Finally, we say that **MLS** is the category with stable calculi as objects and compatible consequence relations as morphisisms.

MLS and stably compact spaces

Theorem (Kegelmann, [3])

MLS *is equivalent to the category* $\mathbf{SCSp}^*$ *of stably compact spaces and compact saturated binary relations.*

Hyperdoctrines: classical picture

$$H : \mathbf{C}^{\mathbf{op}} \longrightarrow \mathbf{Pos}$$

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Hyperdoctrines: classical picture

$$H : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

$$H : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Bool}$$

$$H : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$$

$$H : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{MLS}$$

Signature

Definition

A first-order *signature* Σ consists of

1. A set Sort_Σ of *sorts*. We denote its elements by symbols such as

$$A_\in \mathrm{Sort}_\Sigma$$

2. A set Fun_Σ of *function symbols* f. Each function symbol is equipped with a finite list of source sorts and a single output sort. We write

$$f : A_1 \dots A_n \to B$$

   to express that $A_1, \dots, A_n$ is the list of source sorts of f and B is the output sort. The number n of source sorts of f is called the *arity* of f. If $n = 0$, f is called a *constant* of sort B. We will denote constants as c.

Signature

1. A set Rel_Σ of *relation symbols* R. Each relation symbol is equipped with a finite list of sorts. We write

$$R \dashrightarrow A_1 \dots A_n$$

to express that $A_1 \dots A_n$ is the list of sorts associated with R. The number n of sorts associated with R is called the *arity* of R.

We assume that, for each $A \in \mathrm{Sort}_\Sigma$, there is a denumerable set V_A whose elements are called 'variables' of type A.

Contexts

Contexts $(\Gamma, \Delta, \Theta, \dots)$ are lists of pairs of variables with types [2]. We give a context-free grammar as follows:

$$\Gamma \quad ::= \quad \mathbf{1} \mid \Gamma, x : A$$

where $\mathbf{1}$ stands for the empty context. Remember that for every $A \in \mathsf{Sor}_\Sigma$, there is a set V_A of elements that we called variables of type A. These sets are disjoint, that is, $V_A \cap V_B = \varnothing$ if $A \neq B$.

Contexts

It is usually said that variables in contexts are distinct. Let us make this statement precise. Let P be a set of pairs (Γ, V) where Γ is a context and $V = (V_A)_{A \in \mathrm{Sort}_\Sigma}$ is a family of sets such that, for each $A \in \mathrm{Sort}_\Sigma$, V_A is the set of variables of type A that do not occur in Γ. Let $F : P \to \mathscr{P}(P)$ be a function sending pairs (Γ, V) to a set of pairs (Γ', V') such that.

$$F(\Gamma, V) = \{((\Gamma, x : A),\ V \setminus \{x\}) \mid A \in \mathrm{Sort}_\Sigma,\ x \in V_A\}\,,$$

where $\Gamma, x : A$ is the context obtained by extending Γ with the variable x of type A, and $V \setminus \{x\}$ is the family of sets of variables obtained by removing x from V_A and leaving any other component unchanged.

Contexts

Now, in order to stratify contexts according to their length, we construct them inductively by stages. Formally speaking, for every n, the set of formal contexts of length n is defined inductively as follows .

$$\mathbf{Ctx}^0 = \{(\mathbf{1}, V)\} \qquad \text{where } \; V = (V_A)_{A \in \mathsf{Sort}_\Sigma} \text{ for all } A \in \mathsf{Sort}_\Sigma$$

$$\mathbf{Ctx}^{n+1} = \bigcup_{(\Gamma, V) \in \mathbf{Ctx}^n} F(\Gamma, V)$$

$$\mathbf{Ctx} = \bigcup_{n<\omega} \mathbf{Ctx}^n$$

Notice that because of **1** has no variables, then for all $V_A \in (V_A)_{A \in \mathsf{Sort}_\Sigma}$, V_A has all variables available. Let us say that, for any set of pairs $F(\Gamma, V)$, the length of Γ is the number n of variables in Γ removed from V.

Although contexts are formally represented by pairs (Γ, V), the family V is uniquely determined by the variables occurring in Γ. Hence, from now on, we suppress V and write simply Γ.

Terms

Definition

Let Σ be a signature. For any context Γ, we define the collection $\mathbb{T}_{\Gamma,A}$ of *terms* of type A in context Γ over Σ inductively. We will use $\Gamma \vdash t : A$ to denote that t is a term of type A in context Γ.

1. if $x : A$ occurs in Γ, then $x \in \mathbb{T}_{\Gamma,A}$.
2. if $\Gamma \vdash t_1 : A_1, \ldots, \Gamma \vdash t_n : A_n$ and $f : A_1, \ldots, A_n \to B$ is a function symbol, $f(t_1, \ldots, t_n) \in \mathbb{T}_{\Gamma,B}$.

Context Morphisms

Let us define the set $\mathbf{Ctx}((\Delta, W), (\Gamma, V))$ of substitutions σ inductively on Γ as follows:

- For the empty context,

$$\mathbf{Ctx}((\Delta, W), (\mathbf{1}, V)) = \{[\,]\} \text{ where } V = (V_A)_{A \in \mathsf{Sort}_\Sigma}.$$

- For an extended context $\Gamma, x : A$,
  $\mathbf{Ctx}((\Delta, W), ((\Gamma, x : A), V \setminus \{x\})) = \{\, \sigma ^\frown (x := t) \mid \sigma \in \mathbf{Ctx}(\Delta, \Gamma),\ t \in \mathbb{T}_{\Delta, A}\}$

- $[\,]^*(u) = u \qquad$ where $\mathbf{1} \vdash u : A$
- $(\sigma^\frown(x := t))^*(x) = t$
- $(\sigma^\frown(x := t))^*(y) = \sigma^*(y) \qquad y \neq x.$
- $(\sigma^\frown(x := t))^*(f(t_1, \ldots, t_n)) = f((\sigma^\frown(x := t))^*(t_1), \ldots, (\sigma^\frown(x := t))^*(t_n))$

Identities

- $\mathrm{id}_1 := \;[\,]$
- $\mathrm{id}_{\Gamma, x:A} := \mathrm{id}_\Gamma \frown (x := x)$

Composition

Context morphisms are composable. That is, if $\Theta \xrightarrow{\tau} \Delta \xrightarrow{\sigma} \Gamma$, then $\sigma \circ \tau : \Theta \longrightarrow \Gamma$.
The arrow $\sigma \circ \tau$ is defined inductively as follows:

- $[\,] \circ \tau = [\,]$
- $(\sigma^\frown(x := t)) \circ \tau = (\sigma \circ \tau)^\frown(x := \tau^*(t))$

Let $\sigma : \Delta \longrightarrow \Gamma$ be a context morphism. Let us define $\sigma^+ : \Delta, x : A \longrightarrow \Gamma$ recursively as follows:

- $[\,]^+ = [\,]$
- $(\rho^\frown(y := t))^+ = (\rho^+)^\frown(y := t)$

where, on the right-hand side, t, originally a term in $\mathbb{T}_{\Delta,B}$ is regarded as a term in $\mathbb{T}_{\Delta,x:A,B}$ by weakening.
The extension of σ by the variable $x : A$ is the context morphism

- $\bar{\sigma} := (\sigma^+)^\frown(x := x) : \Delta, x : A \longrightarrow \Gamma, x : A$

Contexts

Theorem (Fundamental Substitution Theorem)

For any composition $\sigma \circ \tau$ and any term u,
$(\sigma \circ \tau)^*(u) = \tau^*(\sigma^*(u))$.

Contexts

Lemma

Composition of context morphisms is associative.

Finally, let us denote the category of contexts as **Ctx**. **Ctx** consists of the following:

- *Objects*: $\Gamma, \Delta, \Theta, \ldots$
- *Arrows*: For any $\Gamma = \{x_1 : A_1, \ldots, x_n : A_n\}$ and $\Delta = \{y_1 : B_1, \ldots, y_m : B_m\}$, arrows $\sigma : \Delta \to \Gamma$ are n-tuples of terms in context $\Delta = \{y_1 : B_1, \ldots, y_m : B_m\}$.

Binary products in **Ctx**

Let Γ and Δ be two objects in **Ctx**. We cannot always define their product by simply concatenating Γ and Δ, since the same variable may occur in both contexts. However, we can choose a fresh copy Δ' of Δ whose variables do not occur in Γ.

Suppose that $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ and
$\Delta = (y_1 : B_1, \ldots, y_m : B_m)$.

Let $\Delta' = (y'_1 : B_1, \ldots, y'_m : B_m)$, where the variables $y'_1, \ldots, y'_m$ are pairwise distinct and none of them occurs in Γ.

We define

$$\Gamma \times \Delta := \Gamma, \Delta'.$$

Binary products

The first projection $\pi_1 : \Gamma, \Delta' \longrightarrow \Gamma$ is defined by

$$\pi_1 := (x_1 := x_1, \ldots, x_n := x_n).$$

The second projection $\pi_2 : \Gamma, \Delta' \longrightarrow \Delta$ is defined by

$$\pi_2 := (y_1 := y'_1, \ldots, y_m := y'_m).$$

Notice that the codomain of π_2 is the original context Δ, while the terms assigned by π_2 are the fresh variables occurring in Δ'.

Binary products

Proposition

The context Γ, Δ', together with π_1 and π_2, is a product of Γ and Δ.

Formulas in Context

For each context Γ, let L_Γ be defined inductively by

$$\phi \in L_\Gamma \quad ::= \quad P_\Gamma \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \forall x : A\, \phi \mid \exists x : A\, \phi,$$

where in the quantified cases

$$\phi \in L_{\Gamma, x:A}.$$

Let $\sigma : \Delta \longrightarrow \Gamma$ be a context morphism. We define the action of σ on formulas recursively as follows.

- $\sigma^*(R(t_1, \ldots, t_n)) = R(\sigma^*(t_1), \ldots, \sigma^*(t_n))$
- $\sigma^*(s = t) = \sigma^*(s) = \sigma^*(t)$
- $\sigma^*(\top) = \top$
- $\sigma^*(\bot) = \bot$
- $\sigma^*(\phi \wedge \psi) = \sigma^*(\phi) \wedge \sigma^*(\psi)$
- $\sigma^*(\phi \vee \psi) = \sigma^*(\phi) \vee \sigma^*(\psi)$
- $\sigma^*(\forall x{:}A\, \phi) = \forall x{:}A\, \bar{\sigma}^*(\phi)$
- $\sigma^*(\exists x{:}A\, \phi) = \exists x{:}A\, \bar{\sigma}^*(\phi)$

In the quantified clauses, $\bar{\sigma} : \Delta, x : A \longrightarrow \Gamma, x : A$ is the extension of σ by the variable $x : A$, defined by $\bar{\sigma} = (\sigma^{+x})^\frown(x := x)$. Thus, $\bar{\sigma}$ acts as σ on the variables of Γ and leaves the newly added variable x unchanged.

Context-Indexed Positive MLS Fibers

Definition

Let $\Vdash_{L_\Gamma}$ be the *smallest relation* between finite sets of formulas in L_Γ such that, for all $\phi, \psi \in L_\Gamma$ and all finite sets $\mathcal{S}, \mathcal{T}$ of formulas in L_Γ, the following rules hold:

$$\frac{}{\phi \Vdash_{L_\Gamma} \phi}$$

$$\frac{}{\bot \Vdash_{L_\Gamma}} \qquad \frac{}{\Vdash_{L_\Gamma} \top}$$

$$\frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \bot} \qquad \frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\top, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}$$

Context-Indexed Positive MLS Fibers

$$\frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \bot} \qquad \frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\top, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}$$

$$\frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi \qquad \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \psi}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi \wedge \psi} \qquad \frac{\phi, \psi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\phi \wedge \psi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}$$

$$\frac{\phi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T} \qquad \psi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\phi \vee \psi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}} \qquad \frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi, \psi}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi \vee \psi}$$

Context-Indexed Positive MLS Fibers

$$\frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi} \qquad \frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}{\phi, \mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}}$$

$$\frac{\mathcal{S}, \phi, \phi, \mathcal{X} \Vdash_{L_\Gamma} \mathcal{T}}{\mathcal{S}, \phi, \mathcal{X} \Vdash_{L_\Gamma} \mathcal{T}} \qquad \frac{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi, \phi, \mathcal{Y}}{\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}, \phi, \mathcal{Y}}$$

Context-Indexed Positive MLS fibers

$$\frac{\mathcal{S}_0 \Vdash_{L_\Gamma} \mathcal{T}_0, \phi \qquad \phi, \mathcal{S}_1 \Vdash_{L_\Gamma} \mathcal{T}_1}{\mathcal{S}_0, \mathcal{S}_1 \Vdash_{L_\Gamma} \mathcal{T}_0, \mathcal{T}_1}$$

Uppercase script letters $\mathcal{S}, \mathcal{T}, \mathcal{X}, \mathcal{Y}$ denote finite sets of formulas in L_Γ.

MLS fibers

Proposition

The smallest relations $\Vdash_{L_\Gamma}$ between finite sets of formulas of L_Γ, for any Γ, induces an MLS-object, that is, a stable continuous calculus.

Proof.

To prove the proposition, we show:

1. $\Vdash_{L_\Gamma}$ is closed under $\textsc{cut}$
2. $\Vdash_{L_\Gamma}$ enjoys sequential cut decomposition
3. the rules for $\wedge$ and $\vee$ are reversible

1) is immediate from the definition.
2) is trivial since we have identity.
3) is proved by induction on derivations. $\square$

Is all of this functorial?

Every context morphism $\sigma : \Delta \longrightarrow \Gamma$ induces, by recursion on formulas, a reindexing map $\sigma^* : L_\Gamma \longrightarrow L_\Delta$.
We define a relation between finite subsets of L_Γ and L_Δ by $\mathcal{S} \vdash_\sigma \mathcal{T}$ iff $\sigma^*(\mathcal{S}) \Vdash_{L_\Delta} \mathcal{T}$.

Lemma

Let $\sigma^* : L_\Gamma \longrightarrow L_\Delta$ be an induced map between two languages indexed by contexts Γ and Δ. Then if $\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}$, then $\sigma^*(\mathcal{S}) \Vdash_{L_\Delta} \sigma^*(\mathcal{T})$.

Is all of this functorial?

Lemma

$\vdash_\sigma$ *is a compatible consequence relation.*

Proof.

Suppose that $\mathcal{S} \Vdash_{L_\Gamma} \mathcal{U}$ and $\mathcal{U} \vdash_\sigma \mathcal{T}$. Then $\sigma^*(\mathcal{U}) \Vdash_{L_\Delta} \mathcal{T}$. By the previous lemma, $\sigma^*(\mathcal{S}) \Vdash_{L_\Delta} \sigma^*(\mathcal{U})$. By cut, $\sigma^*(\mathcal{S}) \Vdash_{L_\Delta} \mathcal{T}$. Then $\mathcal{S} \vdash_\sigma \mathcal{T}$. The other inclusion is similar. $\square$

Is all of this functorial?

For the identity just consider the case in which $\sigma = \mathrm{id}_\Gamma$. By definition, $\mathcal{S} \vdash_{\mathrm{id}_\Gamma} \mathcal{T}$ iff $\mathrm{id}_\Gamma^*(\mathcal{S}) \Vdash_{L_\Gamma} \mathcal{T}$. Then $\mathcal{S} \Vdash_{L_\Gamma} \mathcal{T}$. Then $\vdash_{\mathrm{id}_\Gamma} = \Vdash_{L_\Gamma}$.

Is all of this functorial?

For composition, we want a proof that the following holds:

$$\vdash_{\sigma\circ\tau} = \vdash_\sigma ; \vdash_\tau$$

The strategy is as follows.
Consider finite sets $\mathcal{S} \subseteq L_\Gamma$ and $\mathcal{T} \subseteq L_\Theta$. Assume thar $\mathcal{S} \vdash_\sigma ; \vdash_\tau \mathcal{T}$.
By CUT', there is α such that $\mathcal{S} \vdash_\sigma \alpha$ and $\alpha \vdash_\tau \mathcal{T}$. Then $\sigma^*(\mathcal{S}) \Vdash_{L_\Delta} \alpha$ and $\tau^*(\alpha) \Vdash_{L_\Theta} \mathcal{T}$. By the preservation lemma, $\tau^*(\sigma^*(\mathcal{S})) \Vdash_{L_\Theta} \tau^*(\alpha)$. By the internal CUT, $\tau^*(\sigma^*(\mathcal{S})) \Vdash_{L_\Theta} \mathcal{T}$.
Finally, by composition of substitutions, $(\sigma \circ \tau)^*(\mathcal{S})) \Vdash_{L_\Theta} \mathcal{T}$.
Therefore, $\mathcal{S} \vdash_{\sigma\circ\tau} \mathcal{T}$.

Is all of this functorial?

We just showed that we have a contravariant functor

$$\mathcal{L} : \mathbf{C}^{\mathbf{op}} \longrightarrow \mathbf{MLS}$$

Is all of this functorial?

We just showed that we have a contravariant functor

$$\mathcal{L} : \mathbf{C}^{\mathbf{op}} \longrightarrow \mathbf{MLS}$$

This is not yet enough to obtain a hyperdoctrine, though!

The Beck–Chevalley square

Let $\sigma : \Delta \longrightarrow \Gamma$ be a context morphism, and let $x : A$ be fresh for both Δ and Γ. Consider the square

$$\begin{tikzcd}
\Delta, x{:}A \arrow[r, "\bar{\sigma}"] \arrow[d, "\rho_\Delta"'] & \Gamma, x{:}A \arrow[d, "\rho_\Gamma"] \\
\Delta \arrow[r, "\sigma"'] & \Gamma.
\end{tikzcd}$$

Here,

$$\bar{\sigma} = (\sigma^{+x})^\frown(x := x) : \Delta, x : A \longrightarrow \Gamma, x : A$$

acts as σ on the variables of Γ and leaves the newly added variable x unchanged.

The square commutes: $\rho_\Gamma \circ \bar{\sigma} = \sigma \circ \rho_\Delta$.

Indeed, along the upper-right path, $\bar{\sigma}$ acts as σ on the variables of Γ and leaves x unchanged; then ρ_Γ forgets x. Along the lower-left path, ρ_Δ first forgets x, and σ is then applied. Thus, $\rho_\Gamma \circ \bar{\sigma} = \sigma \circ \rho_\Delta$.

$$\begin{tikzcd}
(L_{\Delta,xA}, \Vdash_{L_{\Delta,xA}}) & (L_{\Gamma,xA}, \Vdash_{L_{\Gamma,xA}}) \arrow[l, "\vdash_{\bar{\sigma}}"'] \\
(L_{\Delta}, \Vdash_{L_{\Delta}}) \arrow[u, "\vdash_{\rho_\Delta}"] & (L_{\Gamma}, \Vdash_{L_{\Gamma}}) \arrow[l, "\vdash_{\sigma}"'] \arrow[u, "\vdash_{\rho_\Gamma}"']
\end{tikzcd}$$

Quantifiers without Quantifier Rules

The starting point is not a syntactic quantifier.
The starting point is reindexing along the projection

$$\rho_\Gamma : \Gamma, x : A \longrightarrow \Gamma.$$

This gives a map

$$\rho_\Gamma^* : L_\Gamma \longrightarrow L_{\Gamma, x:A}.$$

In **MLS**, this map induces consequence relations. The question is whether these relations have adjoints.

The Idea

The construction is syntax-free.
We first look for adjoint pairs in **MLS**:

$$P_\rho \dashv\vdash_{\rho_\Gamma} \dashv Q_\rho .$$

At this point, P_ρ and Q_ρ are just relations.
They are not yet called existential or universal quantifiers.
Their names are determined later, by comparing their behaviour with Lawvere's adjoint characterization of quantifiers.

The Right Adjoint

Suppose we want a right adjoint to the reindexing relation.
We look for a relation

$$Q_\rho : L_{\Gamma,x:A} \longrightarrow L_\Gamma$$

such that

$$\vdash_{\rho_\Gamma} \dashv Q_\rho .$$

This means

$$\Vdash_{L_\Gamma} \subseteq \vdash_{\rho_\Gamma} ; Q_\rho$$

and

$$Q_\rho ; \vdash_{\rho_\Gamma} \subseteq \Vdash_{L_{\Gamma,x:A}} .$$

The second condition says that Q_ρ is below the identity after composing back with reindexing.

The Greatest Relation

The right adjoint, if it exists, is the greatest relation Q such that

$$Q; \vdash_{\rho_\Gamma} \subseteq \Vdash_{L_{\Gamma,x:A}} .$$

In words:

Q is allowed to relate as many pairs as possible,

but after composing with reindexing it must not produce more consequences than the internal consequence relation of $L_{\Gamma,x:A}$. This is the "largest below the identity" idea.

The Left Adjoint

Dually, a left adjoint is the least relation P such that

$$\Vdash_{L_\Gamma} \subseteq P; \vdash_{\rho_\Gamma} .$$

In words:

P is the smallest relation strong enough to recover the identity after composing with reindexing.

Thus the two adjoints are characterized order-theoretically:

left adjoint = least relation satisfying the unit,

right adjoint = greatest relation satisfying the counit.

Why This Gives Quantifiers

Only after the adjoints are constructed do we identify them logically.
Lawvere's characterization says that quantifiers are adjoints to reindexing:

$$\exists_{\rho_\Gamma} \dashv \rho_\Gamma^* \dashv \forall_{\rho_\Gamma}.$$

So if the adjoint relation satisfies the existential law, we call it existential.
If it satisfies the universal law, we call it universal.
The quantifier is not introduced by a rule. It is recognized by its adjoint behaviour.

Final comments

The construction has two stages.

- First, construct adjoint relations in **MLS** using only consequence, composition, and the order by inclusion.
- Then, identify those adjoints as quantifiers by comparing them with Lawvere's adjoint characterization.

Thus, quantification is described syntax-free:

quantifiers are adjoints to reindexing in **MLS**.

Sketch of the proof

Let

$$h = \rho_\Gamma^* : L_\Gamma \longrightarrow L_{\Gamma, x:A}.$$

The map h induces a relation

$$\vdash_h : L_\Gamma \longrightarrow L_{\Gamma, x:A}$$

defined by

$$\mathcal{S} \vdash_h \mathcal{T} \quad \text{if and only if} \quad h(\mathcal{S}) \Vdash_{L_{\Gamma, x:A}} \mathcal{T}.$$

We also define a relation in the opposite direction

$$\vdash^h : L_{\Gamma, x:A} \longrightarrow L_\Gamma$$

by

$$\mathcal{S} \vdash^h \mathcal{T} \quad \text{if and only if} \quad \mathcal{S} \Vdash_{L_{\Gamma, x:A}} h(\mathcal{T}).$$

Below the Identity

Suppose that

$$\mathcal{S}(\vdash^{h};\vdash_{h})\mathcal{T}.$$

By the definition of composition in **MLS**, there is some $\alpha \in L_\Gamma$ such that

$$\mathcal{S} \vdash^{h} \alpha \qquad \text{and} \qquad \alpha \vdash_{h} \mathcal{T}.$$

By definition of the two relations,

$$\mathcal{S} \Vdash_{L_{\Gamma,x:A}} h(\alpha) \qquad \text{and} \qquad h(\alpha) \Vdash_{L_{\Gamma,x:A}} \mathcal{T}.$$

By cut,

$$\mathcal{S} \Vdash_{L_{\Gamma,x:A}} \mathcal{T}.$$

Therefore,

$$\vdash^{h};\vdash_{h} \subseteq \Vdash_{L_{\Gamma,x:A}}.$$

Further work

1. Explore the double-category perspective. Since hyperdoctrines involve different kinds of morphisms, double categories may provide a fruitful framework for dealing with this kind of logics.
2. A natural next step is to study probabilistic powerdomains in the setting of stably compact spaces [5].
   - The probabilistic powerdomain assigns to a space a space of probability valuations.
   - Valuations measure open sets.
   - For stably compact spaces, the space of probability valuations again carries a natural stably compact topology.
   - This suggests a possible probabilistic extension of MLS: formulas describe open sets, while valuations assign probabilistic weights to them.

Thank you!

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