On varieties of modular lattices with complementation

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Outline

1. Lattices with complementation; the $\mathbf{M}_n$'s
2. Congruence properties of $\mathscr{M}$ and $\mathscr{M}_{DM}$
3. Some subvarieties of $\mathscr{M}_{DM}$
4. Axiomatization of $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$
5. Free algebras in $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

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A lattice with complementation is a complemented lattice with a fixed complementation $x \mapsto x'$ (as a unary operation). Formally: an algebra $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ such that …

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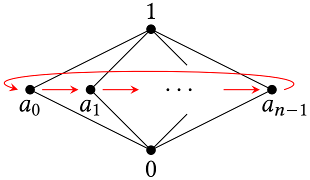
Example: $\mathbf{M}_n = (M_n, \vee, \wedge, ', 0, 1)$ for every integer $n \geq 2$



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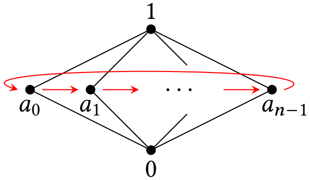

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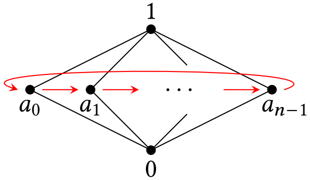

Nice properties:

- $\mathrm{Sg}^{\mathbf{M}_n}(a_i) = \mathbf{M}_n$ for every a_i
- $\mathrm{Con}\,\mathbf{M}_n = \{\Delta, \nabla\}$ for $n \geq 3$

Outline

Congruence properties of $\mathscr{M}$

The variety $\mathscr{M}$ of modular lattices with complementation is

- arithmetical (congruence distributive and permutable),

Consider the term

$$m(x, y, z) = (x \wedge y \wedge z) \vee \big(x \wedge (x \wedge y)'\big) \vee \big(z \wedge (z \wedge y)'\big).$$

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The variety $\mathscr{M}$ of modular lattices with complementation is

- arithmetical (congruence distributive and permutable), and
- regular (every congruence θ is determined by each of its blocks $[a]_\theta$).

Consider the terms

$$t_1(x, y, z) = (x \oplus y) \vee z \quad \text{and} \quad t_2(x, y, z) = (x \oplus y)' \wedge z,$$

where $\oplus$ is any 'symmetric difference' for $\mathscr{M}$, i.e., any binary term such that, in $\mathscr{M}$,

$$x \oplus y = y \oplus x, \qquad x \oplus 0 = x, \quad x \oplus 1 = x', \quad \text{and} \quad x \oplus y = 0 \iff x = y;$$

for instance, $x \oplus y$ could be

$$(x \vee y) \wedge (x \wedge y)' \quad \text{or} \quad (x \vee y) \wedge (x' \vee y'), \quad \text{but not} \quad (x \wedge y') \vee (x' \wedge y).$$

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For every $\mathbf{L} \in \mathscr{M}_{DM}$,

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The variety $\mathscr{M}_{DM}$ is ideal determined; the terms

$$t_1(x, y_1, y_2) = x \wedge (y_1 \vee y_2) \quad \text{and} \quad t_2(x_1, x_2, y) = x_1 \wedge (x_2 \vee y) \wedge (x_1 \wedge x_2)'$$

form a basis of ideal terms for $\mathscr{M}_{DM}$.

Outline

Number of subvarieties of $\mathscr{M}_{DM}$

The lattices M_n ($n \geq 3$) generate only countably many varieties of modular lattices:

$$\mathrm{V}(M_3) \subset \mathrm{V}(M_4) \subset \cdots \subset \mathrm{V}(M_\omega).$$

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For every $\varnothing \neq S \subseteq \{3, 4, 5, \dots\}$, let $\mathcal{S} = \mathrm{V}(\mathbf{M}_s \mid s \in S)$. The map $S \mapsto \mathcal{S}$ is injective:

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Suppose $\mathbf{M}_n \in \mathcal{S}$ but $n \notin S$, for some $n \geq 3$. Then $\mathbf{M}_n \in \mathrm{HSP_U}(\mathbf{M}_s \mid s \in S)$, i.e., $\mathbf{M}_n \in \mathrm{H}(\mathbf{K})$ where $\mathbf{K} \in \mathrm{S}(\mathbf{L})$ where $\mathbf{L} \in \mathrm{P_U}(\mathbf{M}_s \mid s \in S)$. Neither $\mathbf{K}$ nor $\mathbf{L}$ is a boolean algebra. Since the M_s are lattices of length 2 and width ≥ 3, so is L. Hence $L \cong M_\lambda$ for some $\lambda \geq 3$. Since $\mathbf{K}$ is not a boolean algebra, $K \cong M_\kappa$ for some $\kappa \geq 3$. Then $\mathbf{K}$ is a simple algebra, so $\mathbf{M}_n \in \mathrm{H}(\mathbf{K})$ must be isomorphic to $\mathbf{K} \in \mathrm{S}(\mathbf{L})$. However, since $n \notin S$, none of the $\mathbf{M}_s$ has a subalgebra isomorphic to $\mathbf{M}_n$, and hence neither does $\mathbf{L}$; a contradiction.

Some subvarieties of $\mathscr{M}_{DM}$

$$
\begin{tikzcd}[column sep=small]
 & \mathscr{M}_{DM} & & & \\
 & & & & \\
\mathrm{V}(\mathbf{M}_3) & \mathrm{V}(\mathbf{M}_4) & \cdots & \mathrm{V}(\mathbf{M}_n) & \cdots \\
 & \mathscr{B} \arrow[ul, dash] \arrow[u, dash] \arrow[urr, dash] & & & \\
 & \mathscr{T} \arrow[u, dash] & & &
\end{tikzcd}
$$

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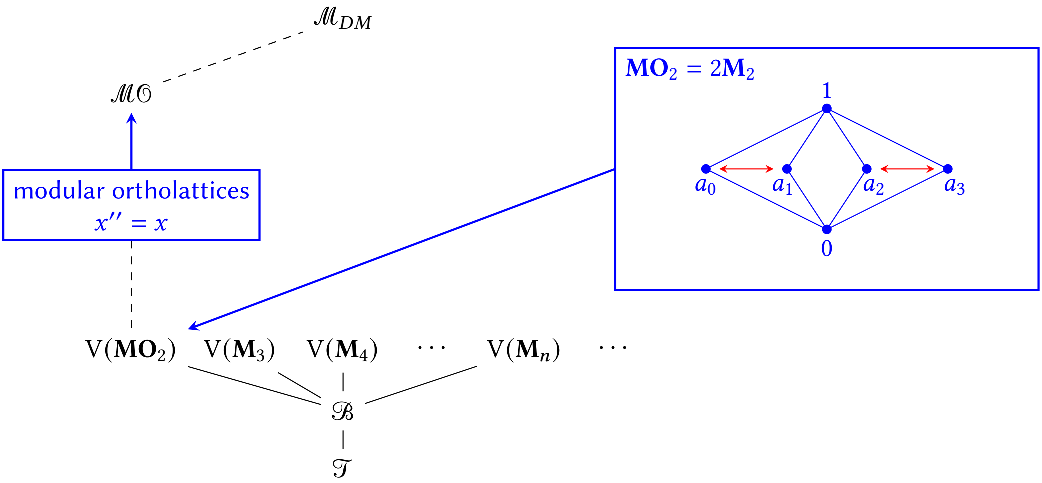

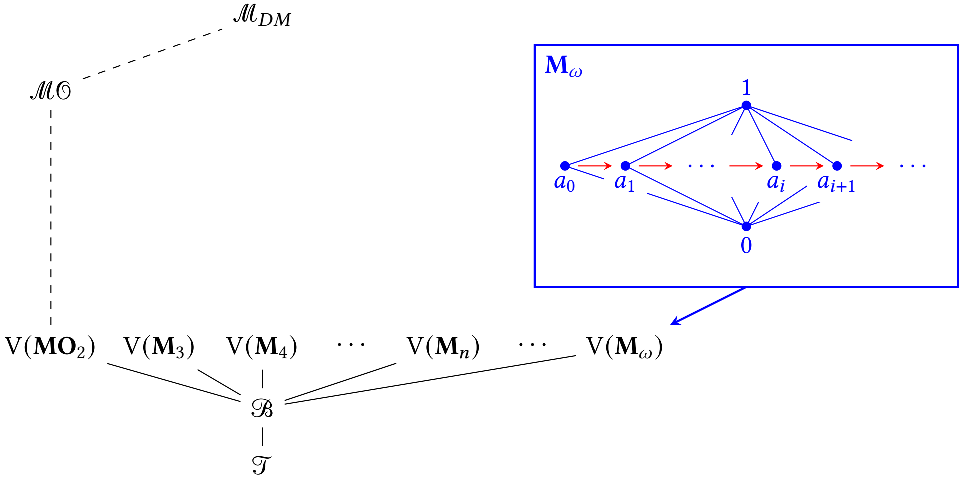
$\mathscr{M}_{DM}$
$\mathscr{MO}$
$V(\mathbf{MO}_2)$
$V(\mathbf{M}_3)$
$V(\mathbf{M}_4)$
$\cdots$
$V(\mathbf{M}_n)$
$\cdots$
$V(\mathbf{M}_\omega)$
$\mathscr{B}$
$\mathscr{T}$
$\mathbf{M}_\omega$
1
a_0
a_1
$\cdots$
a_i
a_{i+1}
$\cdots$
0

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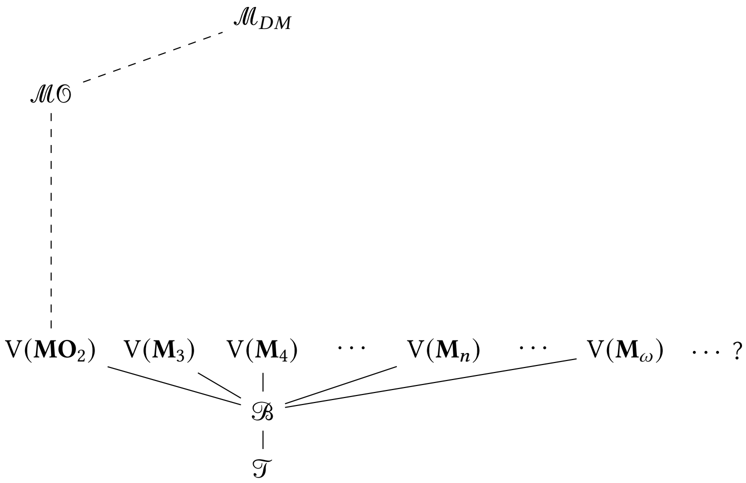

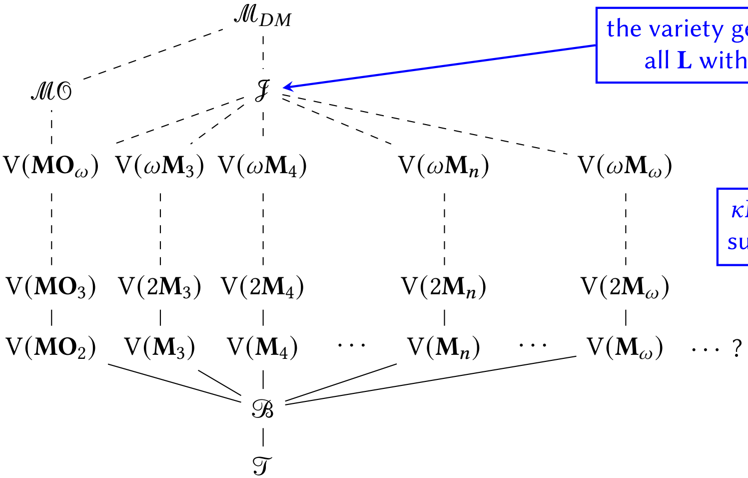

the variety generated by all $\mathbf{L}$ with $L \cong M_\lambda$

$\kappa\mathbf{M}_\lambda$ = the horizontal sum of κ copies of $\mathbf{M}_\lambda$

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JÓNSSON (1968): For every integer $n \geq 3$, the variety generated by the lattice M_n is axiomatized, relative to modular lattices, by the equations

$$u \wedge \big(x \vee (y \wedge z)\big) \wedge (y \vee z) \leq x \vee (u \wedge y) \vee (u \wedge z) \tag{J}$$

and

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i). \tag{M_n}$$

Note: (J) is redundant for $n = 3$.

Axiomatization of $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

For every integer $n \geq 3$, let $\mathscr{J}_n$ be the subvariety of $\mathscr{M}_{DM}$ defined by (J) and (M_n).

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For every integer $n \geq 3$, let $\mathscr{J}_n$ be the subvariety of $\mathscr{M}_{DM}$ defined by (J) and (M_n).

- $\mathscr{J}_3 = \mathrm{V}(\mathbf{M}_3)$.
- If $n \geq 4$, then $\mathscr{J}_n$ is generated by all $\mathbf{L} \in \mathscr{M}_{DM}$ such that $L \cong M_p$ with $3 \leq p \leq n$; hence $\mathrm{V}(\mathbf{M}_3, \ldots, \mathbf{M}_n) \subset \mathscr{J}_n$.

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Suppose $\mathbf{L} \in \mathscr{J}_n \setminus \mathscr{B}$ is subdirectly irreducible. Since $\mathrm{Con}\,\mathbf{L} = \mathrm{Con}\,L$, the lattice L is a subdirectly irreducible member of $\mathrm{V}(M_n)$, and hence $L \cong M_p$ for some $3 \leq p \leq n$.

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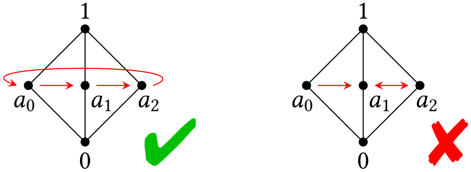
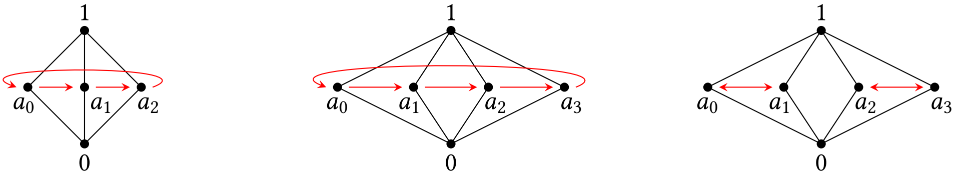
Suppose $\mathbf{L} \in \mathscr{J}_n \setminus \mathscr{B}$ is subdirectly irreducible. Since $\mathrm{Con}\, \mathbf{L} = \mathrm{Con}\, L$, the lattice L is a subdirectly irreducible member of $\mathrm{V}(M_n)$, and hence $L \cong M_p$ for some $3 \leq p \leq n$. For $n = 3$, up to isomorphism, there is only one De Morgan complementation, and so $\mathbf{L} \cong \mathbf{M}_3$:



For example, $\mathcal{J}_4 = \mathrm{V}(\mathbf{M}_3, \mathbf{M}_4, \mathbf{MO}_2) \supset \mathrm{V}(\mathbf{M}_3, \mathbf{M}_4)$:



For every integer $n \geq 3$, we consider the term

$$\tau_n(x) = \bigwedge_{2 \leq k < n} \left(x \vee x^{(k)}\right)$$

where $x^{(k)}$ is a shorthand for $x'' \cdots '$ with k occurrences of $'$. Thus,

$$\begin{aligned} \tau_3(x) &= x \vee x'', \\ \tau_4(x) &= (x \vee x'') \wedge (x \vee x'''), \\ &\vdots \end{aligned}$$

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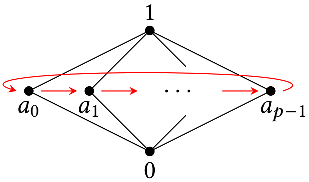
In a lattice L, an element $a \in L$ is neutral if, for all $b, c \in L$, the sublattice generated by $\{a, b, c\}$ is distributive. In a complemented modular lattice, an element is neutral iff it has a unique complement.

In $\mathbf{M}_p$ with $p \geq n$ we have

$$\tau_n(a) = \bigwedge_{2 \leq k < n} \left(a \vee a^{(k)}\right) = \begin{cases} 0 & \text{for } a = 0, \\ 1 & \text{for } a \neq 0. \end{cases}$$



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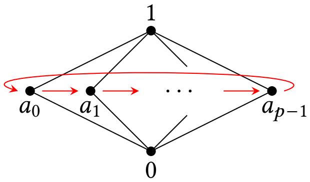
In $\mathbf{M}_p$ with $3 \leq p < n$ we have $\tau_n(a) = a$ for every a.



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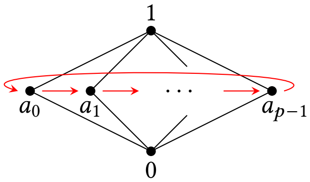

For $\mathbf{M}_p$, the following are equivalent:

- $p \geq n$;
- the elements $\tau_n(a)$ are neutral;
- $\mathbf{M}_p$ satisfies the equation

$$x \wedge (y \vee \tau_n(z)) = (x \wedge y) \vee (x \wedge \tau_n(z)).$$

Axiomatization of $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

For every integer $n \geq 3$, $\mathrm{V}(\mathbf{M}_n)$ is axiomatized, relative to $\mathscr{M}_{DM}$, by the equations

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i) \tag{M_n}$$

and

$$x \wedge (y \vee \tau_n(z)) = (x \wedge y) \vee (x \wedge \tau_n(z)). \tag{T_n}$$

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and

$$x \wedge (y \vee \tau_n(z)) = (x \wedge y) \vee (x \wedge \tau_n(z)). \qquad (\mathrm{T}_n)$$

Suppose that $\mathbf{L} \in \mathscr{M}_{DM} \setminus \mathscr{B}$ satisfies (M_n) and (T_n) and is subdirectly irreducible.

For every integer $n \geq 3$, $V(\mathbf{M}_n)$ is axiomatized, relative to $\mathscr{M}_{DM}$, by the equations

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i) \tag{M_n}$$

and

$$x \wedge (y \vee \tau_n(z)) = (x \wedge y) \vee (x \wedge \tau_n(z)). \tag{T_n}$$

Suppose that $\mathbf{L} \in \mathscr{M}_{DM} \setminus \mathscr{B}$ satisfies (M_n) and (T_n) and is subdirectly irreducible. Then

- $\operatorname{Neutr} L = \{0, 1\}$, since $\operatorname{Con} \mathbf{L} = \operatorname{Con} L$ and $L \cong [0, a] \times [a, 1]$ for any $a \in \operatorname{Neutr} L$. In fact, $\mathbf{L} \cong [\mathbf{0}, \mathbf{a}] \times [\mathbf{a}, \mathbf{1}]$, where $x^\flat = x' \wedge a$ for $x \in [0, a]$ and $x^\sharp = x' \vee a$ for $x \in [a, 1]$.

Axiomatization of $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

- By (T_n), the element $\tau_n(a) = \bigwedge_{2\leq k<n} \left(a \vee a^{(k)}\right)$ is neutral for every a. Hence $\tau_n(a) = 1$ for every $a \neq 0$, so

$$a \vee a' = a \vee a'' = \cdots = a \vee a^{(n-1)} = 1.$$

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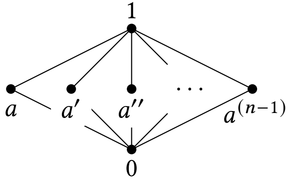
- For every $a \neq 0$, the following copy of M_n is a sublattice of L:



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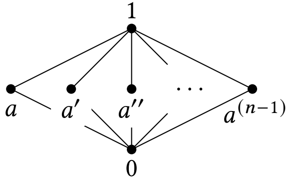

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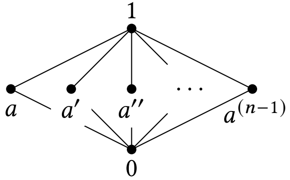

- L is of length 2, hence $L \cong M_\lambda$ for some $\lambda \geq n$.
- By (M_n), $\lambda \leq n$, hence $L \cong M_n$.

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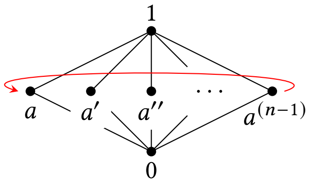

- L is of length 2, hence $L \cong M_\lambda$ for some $\lambda \geq n$.
- By (M_n), $\lambda \leq n$, hence $L \cong M_n$.
- Finally, $a^{(n)} = a$, and so $\mathbf{L} \cong \mathbf{M}_n$.

Axiomatization of $\mathrm{V}(\mathbf{M}_3)$ again

In particular, $\mathrm{V}(\mathbf{M}_3)$ is axiomatized, relative to $\mathscr{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3), \tag{M$_3$}$$

$$x \wedge (y \vee z \vee z'') = (x \wedge y) \vee (x \wedge (z \vee z'')). \tag{T$_3$}$$

We known that $\mathrm{V}(\mathbf{M}_3) = \mathscr{J}_3$, i.e., (T$_3$) is redundant, but it is not straightforward to get (T$_3$) from (M$_3$): If $\mathbf{L} \in \mathscr{J}_3$, then

- for every $a \in L$, $a \in \operatorname{Neutr} L$ iff $a = a''$;
- $\mathbf{L}$ satisfies the equation $x \vee x'' = (x \vee x'')''$;
- hence the elements $a \vee a''$ are neutral, and so $\mathbf{L}$ satisfies (T$_3$).

Without Jónsson's equations

For every integer $n \geq 3$, consider the term

$$\sigma_n(x, y) = \bigwedge_{1 \leq k < n} \left(x \vee y^{(k)}\right).$$

Note that

$$\sigma_n(x, x) = \tau_n(x) = \bigwedge_{2 \leq k < n} \left(x \vee x^{(k)}\right).$$

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- $\mathrm{V}(\mathbf{M}_n)$ is axiomatized by $\sigma_{n+1}(x, y) = x$ and (T_n).

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Relative to $\mathscr{M}_{DM}$,

- $\mathrm{V}(\mathbf{M}_n)$ is axiomatized by $\sigma_{n+1}(x, y) = x$ and (T_n).
- $\mathrm{V}(\omega\mathbf{M}_n)$ is axiomatized by $\tau_{n+1}(x) = x$ and (T_n).

Outline

Free algebras in $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

For every integer $n \geq 3$, the variety $\mathrm{V}(\mathbf{M}_n)$ is a discriminator variety. The term

$$t(x,y,z) = \Big(\big((x \oplus y) \vee (x \oplus y)''\big) \wedge x\Big) \vee \Big(\big((x \oplus y) \vee (x \oplus y)''\big)' \wedge z\Big)$$

is a discriminator term for the algebra $\mathbf{M}_n$, i.e.

$$t(a,b,c) = \begin{cases} a & \text{if } a \neq b, \\ c & \text{if } a = b. \end{cases}$$

Here $\oplus$ is any ‘symmetric difference’ for $\mathscr{M}_{DM}$, e.g., $x \oplus y = (x \vee y) \wedge (x \wedge y)'$.
Only for $n = 3$ we may take $x \oplus y = (x \wedge y') \vee (x' \wedge y)$.

Free algebras in $V(\mathbf{M}_n)$ for $n \geq 3$

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For every integer $n \geq 3$, the automorphism group of $\mathbf{M}_n$ is isomorphic to $\mathbb{Z}_n$.

Free algebras in $\mathrm{V}(\mathbf{M}_n)$ for $n \geq 3$

For any $k \geq 1$ and $n \geq 3$, the free k-generator algebra in $\mathrm{V}(\mathbf{M}_n)$ is

$$\mathbf{F}_{\mathrm{V}(\mathbf{M}_n)}(k) \cong \mathbf{M}_n{}^q \times \mathbf{B}_2{}^r \cong \mathbf{M}_n{}^q \times \mathbf{F}_{\mathscr{B}}(k)$$

where $q = \frac{(n+2)^k - 2^k}{n}$ and $r = 2^k$.

The exponents are

$$\frac{\text{number of valuations of } x_1, \ldots, x_k}{\text{number of automorphisms}}$$

THANK YOU!